Problem Set 11

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In all of the following problems, X is a stratified space with stratification $\mathcal{S}$, and $p : \mathcal{S} \to \mathbb{Z}$ is a perversity function.

1. Prove that $D^b_c(X)$ is a triangulated category. (All the axioms except one are obvious because they hold in $D^b(X)$. The only thing to show is that given a morphism $f : \mathcal{F} \to \mathcal{G}$, you can extend it to a distinguished triangle $\mathcal{F} \to \mathcal{G} \to \mathcal{H} \to \mathcal{F}[1]$. You can of course do that in $D^b(X)$, but is $\mathcal{H}$ necessarily constructible?) (*Hint*: First reduce the problem to the case of one stratum. In that case, be careful: you are dealing with sheaves whose cohomology sheaves are locally constant, but you cannot assume that the sheaves themselves are complexes of locally constant ordinary sheaves.)

2. Let S be a stratum, $\mathcal{E}$ a local system on S, and $\mathcal{F}$ a perverse sheaf with support contained in $\overline{S} \smallsetminus S$. Show that
$$\operatorname{Hom}(\mathrm{IC}(\overline{S}, \mathcal{E}), \mathcal{F}) = \operatorname{Hom}(\mathcal{F}, \mathrm{IC}(\overline{S}, \mathcal{E})) = 0.$$

For the remaining problems, assume that p is a Goresky–MacPherson perversity. That is, $p(S) = \tilde{p}(\dim S)$, where $\tilde{p} : \mathbb{N} \to \mathbb{Z}$ is a function satisfying
$$0 \le \tilde{p}(n) - \tilde{p}(m) \le m - n$$
whenever $n \le m$. Equivalently, $\tilde{p}$ and $\tilde{p}^*$ (where $\tilde{p}^*(n) = -n - \tilde{p}(n)$) are both weakly decreasing.

3. Suppose there is an open dense stratum U in $\mathcal{S}$, and assume that $p(U) < p(S)$ for all strata $S \ne U$. Show that if X is a manifold, then $\mathrm{IC}(\overline{U}, \underline{\mathbb{C}}_U) \simeq \underline{\mathbb{C}}_X[p(U)]$.

4. (a) Suppose $\mathcal{S}$ consists of exactly two strata, U and Z, with $\overline{U} = X$. Assume that $p(U) < p(Z)$. Let $j : U \hookrightarrow X$ be the inclusion map. Show that
$$\mathrm{IC}(\overline{U}, \mathcal{E}) = {}^{\mathrm{std}}\tau_{\le p(Z)-1} Rj_*(\mathcal{E}[\dim U]).$$

   (b) Now, let S be a stratum with the property that $p(S) < p(T)$ for all strata $T \subset \overline{S}$, $T \ne S$. Give a construction of $\mathrm{IC}(\overline{S}, \mathcal{E})$ that uses only derived push-forwards and truncation functors with respect to the *standard* t-structure. (Since the standard truncation functors can be defined by explicit formulas at the level of complexes, this problem shows that intersection cohomology complexes can be defined without using the formalism of t-structures at all.) (*Hint:* Use induction on the number of strata in $\overline{S}$ and imitate part (a).)

5. Let $\mathcal{F}$ be a constructible sheaf. Show that $\mathcal{F} \in {}^pD^b_c(X)^{\le 0}$ if and only if
$$\begin{aligned} \dim \operatorname{supp} H^{\tilde{p}(k)}(\mathcal{F}) &\le k \qquad \text{for all } k, \\ H^i(\mathcal{F}) &= 0 \qquad \text{for all } i > \tilde{p}(0). \end{aligned}$$

   Then show that $\mathcal{F}$ is perverse if and only if the above condition holds for both $\mathcal{F}$ and $\mathbb{D}\mathcal{F}$.