Problem Set 12

April 26, 2007

In all of the following problems, X is a stratified space with stratification $\mathcal{S}$, and X has a unique open stratum U. All perverse sheaves are with respect to the middle perversity.

1. Let $f : Y \to X$ be a semismall resolution. Show that $Rf_*\underline{\mathbb{C}}$ is a perverse sheaf.
2. Let $f : Y \to X$ be a small resolution. Show that $Rf_*\underline{\mathbb{C}} \simeq \mathrm{IC}(X, \underline{\mathbb{C}})$.
3. Suppose X has exactly two strata, U and Z (so of course U is open and Z is closed). Suppose $p(U) > p(Z)$ (in particular, this is *not* a Goresky–MacPherson perversity). Show that all perverse sheaves are of the form $j_!\mathcal{E}[p(U)] \oplus i_*\mathcal{F}[p(Z)]$, where $\mathcal{E}$ and $\mathcal{F}$ are local systems on U and Z, respectively, and $j : U \hookrightarrow X$ and $i : Z \hookrightarrow X$ are the inclusion maps. Thus, there is an equivalence of categories
$$M(X) \overset{\sim}{\longleftrightarrow} \{\text{representations of } \pi_1(U) \times \pi_1(Z)\}.$$
Thus, perverse sheaves with respect to a non-Goresky–MacPherson perversity are not that interesting—they do not encode topological information about the singularities of X.
4. Let $\mathcal{F}$ be a perverse sheaf, and let S be a stratum that is open in the support of $\mathcal{F}$. Show that $(\mathcal{F}|_S)[-p(S)]$ is a local system (in particular, you must show that it is an ordinary sheaf, not a complex of sheaves). Next, let $\mathcal{E}$ be that local sytem. Show that $\mathrm{IC}(\bar{S}, \mathcal{E})$ occurs as a quotient in the Jordan–Hölder series for $\mathcal{F}$. (That is, show that $\mathrm{IC}(\bar{S}, \mathcal{E})$ is a quotient of some sub-perverse sheaf of $\mathcal{F}$.)
5. Let $\mathcal{F}$ and $\mathcal{G}$ be two perverse sheaves. Show that $\mathcal{E}xt^i(\mathcal{F}, \mathcal{G}) = 0$ for all $i < 0$. (Recall that $\mathcal{E}xt^i(\mathcal{F}, \mathcal{G}) = H^i(R\mathcal{H}om(\mathcal{F}, \mathcal{G}))$.) (*Hint:* Use induction on the number of strata and a distinguished triangle associated to open and closed inclusions. It may be useful to recall the facts below.)
$$\begin{aligned} j^{-1}R\mathcal{H}om(\mathcal{F}, \mathcal{G}) &\simeq R\mathcal{H}om(j^{-1}\mathcal{F}, j^{-1}\mathcal{G}) && \text{if } j \text{ is an open inclusion,} \\ f^!R\mathcal{H}om(\mathcal{F}, \mathcal{G}) &\simeq R\mathcal{H}om(f^{-1}\mathcal{F}, f^!\mathcal{G}) && \text{for any map } f. \end{aligned}$$
Note: This result, called "vanishing of negative local Ext's," is an important step in the proof of a theorem that states that perverse sheaves on open sets can be glued together to form a perverse sheaf on the whole space, just like ordinary sheaves can.
6. Show that $\mathrm{Hom}(\mathrm{IC}(\bar{S}, \mathcal{E}), \mathrm{IC}(\bar{S}, \mathcal{F})) \simeq \mathrm{Hom}(\mathcal{E}, \mathcal{F})$ (where $\mathcal{E}$ and $\mathcal{F}$ are local systems on S).
7. (Schur's lemma for perverse sheaves) Show that if $\mathcal{E}$ is a simple local system on a connected stratum S, then $\mathrm{Hom}(\mathrm{IC}(\bar{S}, \mathcal{E}), \mathrm{IC}(\bar{S}, \mathcal{E})) \simeq \mathbb{C}$. (*Hint:* First show that $\mathrm{Hom}(\mathcal{E}, \mathcal{E}) \simeq \mathbb{C}$, and use the preceding exercise.)