Problem Set 2

January 30, 2007

From this problem set on, the following assumptions are in effect:

- All sheaves are sheaves of complex vector spaces unless otherwise specified.
- All topological spaces are locally path-connected, semilocally simply connected, locally compact, second-countable, and Hausdorff. Unless otherwise specified, they are also path-connected.

In problems that ask you to "identify" a sheaf, you should either show that the sheaf is isomorphic to some sheaf we have discussed in class, or give as explicit a description as you can of sections of the sheaf over a typical connected open set.

1. Let $X = \mathbb{C} \smallsetminus \{0\}$, and let $\mathcal{Q}$ be the square-root sheaf on X.
   (a) Let $f : X \to X$ be the map $f(z) = z^2$. Identify $f_*\mathcal{Q}$ and $f^{-1}\mathcal{Q}$.
   (b) Show that $\mathrm{Hom}(\mathcal{Q}, \underline{\mathbb{C}}) = 0$. Identify the sheaf $\mathcal{H}om(\mathcal{Q}, \underline{\mathbb{C}})$. (It's not the zero sheaf.)
   (c) Identify $\mathcal{Q} \otimes \mathcal{Q}$. Also, show explicitly in this example that the presheaf tensor product $\mathcal{Q} \otimes_{\mathrm{ps}} \mathcal{Q}$ is not a sheaf.
   (d) Let $g : X \to X$ be the map $g(z) = z^3$. Identify $g_*\underline{\mathbb{C}}$. (*Hint*: The answer is $\underline{\mathbb{C}} \oplus \mathcal{F} \oplus \mathcal{G}$, where $\mathcal{F}$ is the "cube-root sheaf" given by

   $$\mathcal{F}(U) = \{\mathbb{C}\text{-linear combinations of functions } k : U \to \mathbb{C} \text{ such that } k(z)^3 = z\},$$

   and $\mathcal{G}$ is another locally constant sheaf that you'll have to identify by yourself.)
   (e) Identify $\mathcal{H}om(\mathcal{F}, \underline{\mathbb{C}})$. (*Hint*: The answer is the sheaf $\mathcal{G}$ that appears in the previous question. If you have already answered that question, then you know what $\mathcal{G}$ is, and you just have to show that $\mathcal{H}om(\mathcal{Q}, \underline{\mathbb{C}})$ is isomorphic to it. Otherwise, you can use this problem to help you answer the previous one.)

2. Let $j : U \hookrightarrow X$ be the inclusion of an open set. If $\mathcal{F}$ is a sheaf on U and $\mathcal{G}$ is a sheaf on X, show that $\mathrm{Hom}(j_!\mathcal{F}, \mathcal{G}) \simeq \mathrm{Hom}(\mathcal{F}, j^{-1}\mathcal{G})$.

3. (Hartshorne, Exercise 1.19(c)) Let $U \subset X$ be an open set, let $Z = X \smallsetminus U$ be its complement, and let $j : U \hookrightarrow X$ and $i : Z \hookrightarrow X$ be the inclusion maps. Given a sheaf $\mathcal{F}$ on X, show that there is a short exact sequence of sheaves
   $$0 \to j_!(\mathcal{F}|_U) \to \mathcal{F} \to i_*(\mathcal{F}|_Z) \to 0.$$

4. Let $\mathfrak{Sh}(X)$ be the category of sheaves on X, and let $\mathfrak{PSh}(X)$ be the category of presheaves on X. Let $I : \mathfrak{Sh}(X) \to \mathfrak{PSh}(X)$ be the inclusion functor (note that $\mathfrak{Sh}(X)$ is a subcategory of $\mathfrak{PSh}(X)$). We also have the sheafification functor $^+ : \mathfrak{PSh}(X) \to \mathfrak{Sh}(X)$. Show that $(^+, I)$ is an adjoint pair of functors.