Problem Set 3

February 1, 2007

In this problem set (and henceforth in the course), the following slight abuse of language will be made: if $\rho : \pi_1(X, x_0) \to GL(E)$ is a representation of $\pi_1(X, x_0)$ on the vector space E, we will call E itself "the representation." (Thus, "Let E be a representation of $\pi_1(X, x_0)$" means "Let E be a complex vector space, and suppose there is a representation $\pi_1(X, x_0) \to GL(E)$ of $\pi_1(X, x_0)$ on E.")

1. Let $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ be sheaves of abelian groups on X. Prove that
$$\mathrm{Hom}(\mathcal{F} \otimes \mathcal{G}, \mathcal{H}) \simeq \mathrm{Hom}(\mathcal{F}, \mathcal{H}om(\mathcal{G}, \mathcal{H})) \qquad \text{and} \qquad \mathcal{H}om(\mathcal{F} \otimes \mathcal{G}, \mathcal{H}) \simeq \mathcal{H}om(\mathcal{F}, \mathcal{H}om(\mathcal{G}, \mathcal{H}))$$
by using the corresponding facts for abelian groups.

2. Problem 2 of Problem Set 2 asked you to show that $(j_!, j^{-1})$ is an adjoint pair, where $j : U \hookrightarrow X$ is an open inclusion. State and prove a sheaf-Hom version of that theorem. (Note that it does not make sense to say $\mathcal{H}om_X(j_!\mathcal{F}, \mathcal{G}) \simeq \mathcal{H}om_U(\mathcal{F}, j^{-1}\mathcal{G})$.)

3. Show that there is an equivalence of categories
$$(\text{local systems on } X) \overset{\sim}{\longleftrightarrow} (\text{representations of } \pi_1(X, x_0)).$$
In other words: In class, we have shown that there is a bijection between isomorphism classes of local systems and isomorphism classes of representations of $\pi_1(X, x_0)$. Now, let E and F be representations of $\pi_1(X, x_0)$, and let $\mathcal{E}$ and $\mathcal{F}$ be the corresponding local systems on X. Given a morphism $\phi : \mathcal{E} \to \mathcal{F}$, associate to it an equivariant linear transformation (also called an "intertwining operator") $S : E \to F$. Conversely, given an equivariant linear transformation $T : E \to F$, construct an associated morphism $\psi : \mathcal{E} \to \mathcal{F}$. Finally, show that the two assignments $\phi \mapsto S$ and $T \mapsto \psi$ are inverse to each other.

4. If $\mathcal{E}$ and $\mathcal{F}$ are local systems, show that $\mathcal{H}om(\mathcal{E}, \mathcal{F})$ and $\mathcal{E} \otimes \mathcal{F}$ are as well. What are their ranks in terms of $\operatorname{rank} \mathcal{E}$ and $\operatorname{rank} \mathcal{F}$? (The **rank** of a local system is the dimension of any of its stalks.)

5. In the setting of the previous problem, let E and F be the representations of $\pi_1(X, x_0)$ corresponding to $\mathcal{E}$ and $\mathcal{F}$. There is a natural way to regard the vector spaces $\mathrm{Hom}(E, F)$ and $E \otimes F$ as representations of $\pi_1(X, x_0)$ as well. (Ask me if you can't figure out how yourself.) Show that the local systems corresponding to these representations are precisely $\mathcal{H}om(\mathcal{E}, \mathcal{F})$ and $\mathcal{E} \otimes \mathcal{F}$.

6. Let E and F be two nonisomorphic irreducible representations of $\pi_1(X, x_0)$. (A representation is **irreducible** if contains no nontrivial subspace that is stable under the action of $\pi_1(X, x_0)$.) Show that $\mathrm{Hom}(\mathcal{E}, \mathcal{F}) = 0$. (Note, however, that the sheaf $\mathcal{H}om(\mathcal{E}, \mathcal{F})$ is not, in general, the zero sheaf.)