In this problem set (and henceforth in the course), the following slight abuse of language will be made: if \( \rho : \pi_1(X, x_0) \rightarrow GL(E) \) is a representation of \( \pi_1(X, x_0) \) on the vector space \( E \), we will call \( E \) itself “the representation.” (Thus, “Let \( E \) be a representation of \( \pi_1(X, x_0) \)” means “Let \( E \) be a complex vector space, and suppose there is a representation \( \pi_1(X, x_0) \rightarrow GL(E) \) of \( \pi_1(X, x_0) \) on \( E \).”)

1. Let \( F, G, \) and \( H \) be sheaves of abelian groups on \( X \). Prove that

\[
\text{Hom}(F \otimes G, H) \cong \text{Hom}(F, \text{Hom}(G, H)) \quad \text{and} \quad \text{Hom}(F \otimes G, H) \cong \text{Hom}(F, \text{Hom}(G, H))
\]

by using the corresponding facts for abelian groups.

2. Problem 2 of Problem Set 2 asked you to show that \( (j_!, j_\ast) \) is an adjoint pair, where \( j : U \hookrightarrow X \) is an open inclusion. State and prove a sheaf-Hom version of that theorem. (Note that it does not make sense to say \( \text{Hom}_X(j_!F, G) \cong \text{Hom}_U(F, j_\ast G) \).)

3. Show that there is an equivalence of categories

\[
\text{(local systems on } X) \leftrightarrow \text{(representations of } \pi_1(X, x_0)).
\]

In other words: In class, we have shown that there is a bijection between isomorphism classes of local systems and isomorphism classes of representations of \( \pi_1(X, x_0) \). Now, let \( E \) and \( F \) be representations of \( \pi_1(X, x_0) \), and let \( E \) and \( F \) be the corresponding local systems on \( X \). Given a morphism \( \phi : E \rightarrow F \), associate to it an equivariant linear transformation (also called an “intertwining operator”) \( S : E \rightarrow F \). Conversely, given an equivariant linear transformation \( T : E \rightarrow F \), construct an associated morphism \( \psi : E \rightarrow F \). Finally, show that the two assignments \( \phi \mapsto S \) and \( T \mapsto \psi \) are inverse to each other.

4. If \( E \) and \( F \) are local systems, show that \( \text{Hom}(E, F) \) and \( E \otimes F \) are as well. What are their ranks in terms of rank \( E \) and rank \( F \)? (The rank of a local system is the dimension of any of its stalks.)

5. In the setting of the previous problem, let \( E \) and \( F \) be the representations of \( \pi_1(X, x_0) \) corresponding to \( E \) and \( F \). There is a natural way to regard the vector spaces \( \text{Hom}(E, F) \) and \( E \otimes F \) as representations of \( \pi_1(X, x_0) \) as well. (Ask me if you can’t figure out how yourself.) Show that the local systems corresponding to these representations are precisely \( \text{Hom}(E, F) \) and \( E \otimes F \).

6. Let \( E \) and \( F \) be two nonisomorphic irreducible representations of \( \pi_1(X, x_0) \). (A representation is irreducible if contains no nontrivial subspace that is stable under the action of \( \pi_1(X, x_0) \).) Show that \( \text{Hom}(E, F) = 0 \). (Note, however, that the sheaf \( \text{Hom}(E, F) \) is not, in general, the zero sheaf.)