Problem Set 4

February 8, 2007

1. Let $p : Y \to X$ be a finite-to-one covering map. If $\mathcal{E}$ is a local system on Y, show that $p_*\mathcal{E}$ is a local system on X. If p is not finite-to-one, the proof of the previous statement does not go through, but it can be fixed up by imposing an additional condition on $\mathcal{E}$. Find such a condition, and prove that with this extra condition, $p_*\mathcal{E}$ is again a local system. (*Hint*: If Y is compact, no additional condition on $\mathcal{E}$ is needed.)

2. Recall that there is a one-to-one correspondence between covering spaces over X (up to isomorphism) and subgroups of its fundamental group $\pi_1(X, x_0)$ (up to conjugacy. Suppose $p : Y \to X$ is a covering map corresponding to the subgroup $H \subset \pi_1(X, x_0)$. Let $E = \mathbb{C}[\pi_1(X, x_0)/H]$, the vector space of formal linear combinations of cosets of H with complex coefficients. There is an obvious representation of $\pi_1(X, x_0)$ on E. Show that the local system $p_*\underline{\mathbb{C}}$ is the one corresponding to this representation.

3. Show that the category of sheaves of abelian groups on a fixed topological space X is an abelian category.

4. Let $S : \mathcal{A} \to \mathcal{B}$ and $T : \mathcal{B} \to \mathcal{A}$ be an adjoint pair of functors. Show that S is right-exact and that T is left-exact. (Be sure to do this using the language of categories—you cannot talk about "elements" of kernels and images, because in a general abelian category, the concept of "element" may not make sense.) It immediately follows that the functors

$$\mathrm{Hom}(\mathcal{G}, \cdot), \quad \mathcal{H}om(\mathcal{G}, \cdot), \quad f_*$$

are left-exact (here $\mathcal{G}$ is a sheaf and $f : X \to Y$ is a continuous map), and

$$\cdot \otimes \mathcal{G}, \quad f^{-1}, \quad j_!$$

are right-exact (here $j : U \to X$ is an open inclusion). Finally, note that j^{-1} (again, j is an open inclusion) is exact, because it has adjoints on both sides: j_* is its right adjoint, and $j_!$ is its left adjoint.

5. Show that f^{-1} is actually an exact functor for any continuous map $f : X \to Y$, not just an open inclusion.

6. Show that if $j : U \to X$ is an open inclusion, then $j_!$ is an exact functor. (It suffices to show that it is right-exact.)