Problem Set 5

February 15, 2007

1. Gelfand & Manin, *Methods of Homological Algebra*, Exercise II.5.2 (page 119).

2. *Ibid.*, Exercise II.5.5 (page 120). (*Note*: See Exercise II.5.4 for the definition of "$f(y)$.")

3. Let $A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet \longrightarrow A[1]^\bullet$ be a distinguished triangle (in $K(\mathcal{A})$ or $D(\mathcal{A})$). Show that $g \circ f = 0$.

4. Let $0 \longrightarrow A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet \longrightarrow 0$ be a short exact sequence in $C(\mathcal{A})$. Define a morphism $\theta : \operatorname{cone}^\bullet f \to C^\bullet$ by the matrix $\begin{pmatrix} 0 & g \end{pmatrix}$. Show that θ is indeed a morphism of complexes (*i.e.*, that it commutes with differentials) and that the following diagram commutes:

$$\begin{tikzcd} A^\bullet \arrow[r, "f"] \arrow[d, equal] & B^\bullet \arrow[r] \arrow[d, equal] & \operatorname{cone}^\bullet f \arrow[d, "\theta"] \\ A^\bullet \arrow[r, "f"] & B^\bullet \arrow[r, "g"] & C^\bullet \end{tikzcd}$$

   Finally, show that θ is a quasi-isomorphism. Deduce that in $D(\mathcal{A})$ (but not necessarily in $K(\mathcal{A})$), there is a distinguished triangle of the form $A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet \to A[1]^\bullet$.

5. Prove that the octahedral property holds for distinguished triangles in $D(\mathcal{A})$.

6. Given a complex $A^\bullet$, define new complexes $\tau_{\leq 0}A^\bullet$ and $\tau_{\geq 1}A^\bullet$ by

$$(\tau_{\leq 0}A)^i = \begin{cases} A^i & \text{if } i < 0, \\ \ker d_A^0 & \text{if } i = 0, \\ 0 & \text{if } i > 0 \end{cases} \qquad \text{and} \qquad (\tau_{\geq 1}A)^i = \begin{cases} 0 & \text{if } i < 1 \\ \operatorname{cok} d_A^0 & \text{if } i = 1, \\ A^i & \text{if } i > 1. \end{cases}$$

   There is an obvious way to define differentials (so that $\tau_{\leq 0}A^\bullet$ and $\tau_{\geq 1}A^\bullet$ are actually complexes), as well as obvious morphisms $\tau_{\leq 0}A^\bullet \to A^\bullet$ and $A^\bullet \to \tau_{\geq 1}A^\bullet$. Find all of these, and then show that there is a distinguished triangle

$$\tau_{\leq 0}A^\bullet \to A^\bullet \to \tau_{\geq 1}A^\bullet \to \tau_{\leq 0}A[1]^\bullet.$$

   What can you say about the cohomology objects of the three complexes $\tau_{\leq 0}A^\bullet$, $A^\bullet$, and $\tau_{\geq 1}A^\bullet$?