## Problem Set 6

March 1, 2007

1. Let  $\mathcal{F}^{\bullet}$  and  $\mathcal{G}^{\bullet}$  be complexes of sheaves. Show that  $\underline{\operatorname{Hom}}(\mathcal{F}^{\bullet}, \mathcal{G}^{\bullet})$  and  $\mathcal{F}^{\bullet} \otimes \mathcal{G}^{\bullet}$  (graded tensor product) are well-defined complexes (that is, that  $d^2 = 0$ ). In class, we defined R Hom and  $\otimes^L$  as functors only of the second variable: for a fixed  $\mathcal{F}^{\bullet} \in C^{-}(\mathfrak{Sh}_{X})$ , we have

$$R\operatorname{Hom}(\mathcal{F}^{\bullet}, -): D^{+}(\mathfrak{Sh}_{X}) \to D^{+}(\mathfrak{Ab})$$
$$\mathcal{F}^{\bullet} \overset{L}{\otimes} -: D^{-}(\mathfrak{Sh}_{X}) \to D^{-}(\mathfrak{Sh}_{X})$$

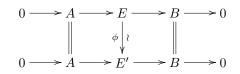
Show that *R* Hom can be regarded as a contravariant functor in its first variable, and that  $\otimes^{L}$  can be regarded as a covariant functors in its first variable. *A priori*, these variables are  $C^{-}(\mathfrak{Sh}_{X})$ ; show that in fact they can be regarded as being in  $D^{-}(\mathfrak{Sh}_{X})$ . Thus, *R* Hom and  $\otimes^{L}$  become bifunctors:

$$R\operatorname{Hom}: D^{-}(\mathfrak{Sh}_{X}) \times D^{+}(\mathfrak{Sh}_{X}) \to D^{+}(\mathfrak{Ab})$$
$$\overset{L}{\otimes}: D^{-}(\mathfrak{Sh}_{X}) \times D^{-}(\mathfrak{Sh}_{X}) \to D^{-}(\mathfrak{Sh}_{X})$$

- 2. Let  $F : \mathcal{A} \to \mathcal{B}$  be a left-exact functor, and suppose that  $\mathcal{A}$  has enough injectives, so that RF is defined. Given an object A of  $\mathcal{A}$ , regard it as a complex with nonzero term only in degree 0 in the obvious way. Show that  $H^0(RF(A)) \simeq F(A)$ .
- 3. Let  $F : \mathcal{A} \to \mathcal{B}$  be a left-exact functor, and suppose that  $\mathcal{A}$  has enough injectives, so that RF is defined. If F happens to actually be exact, it gives rise to a functor  $F : D(\mathcal{A}) \to D(\mathcal{B})$  in a more direct way than the derived-functor construction. Show that this latter functor coincides with RF on  $D^+(\mathcal{A})$ .
- 4. Let  $A^{\bullet}$  and  $B^{\bullet}$  be complexes of objects of an abelian category  $\mathcal{A}$  with enough injectives. Show that  $H^0(R\operatorname{Hom}(A^{\bullet}, B^{\bullet})) \simeq \operatorname{Hom}_{D(\mathcal{A})}(A^{\bullet}, B^{\bullet}).$
- 5. (Extensions) Let A and B be objects of an abelian category  $\mathcal{A}$  that has enough injectives. An *extension* of B by A is simply a short exact sequence

$$0 \to A \to E \to B \to 0.$$

Two extensions are *isomorphic* if there exists an isomorphism  $\phi$  making the following diagram commute:



Show that there is a bijection between isomorphism classes of extensions of B by A and elements of the abelian group  $\text{Ext}^1(B, A)$ . (*Hint*: Regard A and B as complexes, and first show, using distinguished triangles, that there is a bijection between isomorphism classes of extensions and  $\text{Hom}_{D(\mathcal{A})}(B, A[1])$ . Then use the preceding exercise.) What extension corresponds to  $0 \in \text{Ext}^1(B, A)$ ?

6. Let  $\mathcal{F}^{\bullet}$  and  $\mathcal{G}^{\bullet}$  be complexes of sheaves on Y, and let  $f : X \to Y$  be a continuous map. Show that  $f^{-1}(\mathcal{F}^{\bullet} \otimes^{L} \mathcal{G}^{\bullet}) \simeq (f^{-1}\mathcal{F}^{\bullet}) \otimes^{L} (f^{-1}\mathcal{G}^{\bullet})$ . (You'll need to first prove the corresponding statement in the nonderived setting.)