Problem Set 6

March 1, 2007

1. Let $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$ be complexes of sheaves. Show that $\underline{\mathrm{Hom}}(\mathcal{F}^\bullet, \mathcal{G}^\bullet)$ and $\mathcal{F}^\bullet \otimes \mathcal{G}^\bullet$ (graded tensor product) are well-defined complexes (that is, that $d^2 = 0$). In class, we defined $R\mathrm{Hom}$ and $\otimes^L$ as functors only of the second variable: for a fixed $\mathcal{F}^\bullet \in C^-(\mathfrak{Sh}_X)$, we have

$$R\mathrm{Hom}(\mathcal{F}^\bullet, -) : D^+(\mathfrak{Sh}_X) \to D^+(\mathfrak{Ab})$$
$$\mathcal{F}^\bullet \overset{L}{\otimes} - : D^-(\mathfrak{Sh}_X) \to D^-(\mathfrak{Sh}_X)$$

Show that $R\mathrm{Hom}$ can be regarded as a contravariant functor in its first variable, and that $\otimes^L$ can be regarded as a covariant functors in its first variable. *A priori*, these variables are $C^-(\mathfrak{Sh}_X)$; show that in fact they can be regarded as being in $D^-(\mathfrak{Sh}_X)$. Thus, $R\mathrm{Hom}$ and $\otimes^L$ become bifunctors:

$$R\mathrm{Hom} : D^-(\mathfrak{Sh}_X) \times D^+(\mathfrak{Sh}_X) \to D^+(\mathfrak{Ab})$$
$$\overset{L}{\otimes} : D^-(\mathfrak{Sh}_X) \times D^-(\mathfrak{Sh}_X) \to D^-(\mathfrak{Sh}_X)$$

2. Let $F : \mathcal{A} \to \mathcal{B}$ be a left-exact functor, and suppose that $\mathcal{A}$ has enough injectives, so that RF is defined. Given an object A of $\mathcal{A}$, regard it as a complex with nonzero term only in degree 0 in the obvious way. Show that $H^0(RF(A)) \simeq F(A)$.

3. Let $F : \mathcal{A} \to \mathcal{B}$ be a left-exact functor, and suppose that $\mathcal{A}$ has enough injectives, so that RF is defined. If F happens to actually be exact, it gives rise to a functor $F : D(\mathcal{A}) \to D(\mathcal{B})$ in a more direct way than the derived-functor construction. Show that this latter functor coincides with RF on $D^+(\mathcal{A})$.

4. Let $A^\bullet$ and $B^\bullet$ be complexes of objects of an abelian category $\mathcal{A}$ with enough injectives. Show that $H^0(R\mathrm{Hom}(A^\bullet, B^\bullet)) \simeq \mathrm{Hom}_{D(\mathcal{A})}(A^\bullet, B^\bullet)$.

5. (Extensions) Let A and B be objects of an abelian category $\mathcal{A}$ that has enough injectives. An *extension* of B by A is simply a short exact sequence

$$0 \to A \to E \to B \to 0.$$

Two extensions are *isomorphic* if there exists an isomorphism ϕ making the following diagram commute:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d, "\phi"', "\wr"] & B \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & A \arrow[r] & E' \arrow[r] & B \arrow[r] & 0 \end{tikzcd}$$

Show that there is a bijection between isomorphism classes of extensions of B by A and elements of the abelian group $\mathrm{Ext}^1(B, A)$. (*Hint*: Regard A and B as complexes, and first show, using distinguished triangles, that there is a bijection between isomorphism classes of extensions and $\mathrm{Hom}_{D(\mathcal{A})}(B, A[1])$. Then use the preceding exercise.) What extension corresponds to $0 \in \mathrm{Ext}^1(B, A)$?

6. Let $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$ be complexes of sheaves on Y, and let $f : X \to Y$ be a continuous map. Show that $f^{-1}(\mathcal{F}^\bullet \otimes^L \mathcal{G}^\bullet) \simeq (f^{-1}\mathcal{F}^\bullet) \otimes^L (f^{-1}\mathcal{G}^\bullet)$. (You'll need to first prove the corresponding statement in the nonderived setting.)