# Problem Set 7

*March 13, 2007*

1. Let $\mathcal{F}$ be a sheaf on $X$, and let $\mathcal{G}$ be an injective sheaf on $X$. Show that $\mathcal{H}om(\mathcal{F},\mathcal{G})$ is flasque. Your proof should be valid in the category of sheaves of abelian groups (in particular, do not assume that $\mathcal{F}$ is flat, and do not cite the result proved in class that states that $\mathcal{H}om(\mathcal{F},\mathcal{G})$ is injective if $\mathcal{F}$ is flat and $\mathcal{G}$ is injective).

2. Show that a sheaf $\mathcal{F}$ is injective if and only if $\mathcal{H}om(-,\mathcal{F})$ is an exact functor.

3. (Uniqueness of adjoint functors) Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor of abelian categories, and let $G, H : \mathcal{B} \to \mathcal{A}$ be two functors that are both right adjoint to $F$. Show that $G \simeq H$. *I.e.*, show that there is a rule $\eta$ that assigns to every object $B \in \mathcal{B}$ a morphism $\eta(B) : G(B) \to H(B)$ in $\mathcal{A}$ such that for every morphism $f : B \to C$ in $\mathcal{B}$, the following square commutes:

$$\begin{tikzcd} G(B) \arrow[r, "\eta(B)"] \arrow[d, "G(f)"'] & H(B) \arrow[d, "H(f)"] \\ G(C) \arrow[r, "\eta(C)"'] & H(C) \end{tikzcd}$$

4. Let $j : U \hookrightarrow X$ be an inclusion of an open set. Show that $j^! = j^{-1}$.

5. Let $j : U \hookrightarrow X$ be an inclusion of an open set. Let $Z = X \smallsetminus U$, and let $i : Z \hookrightarrow X$ be the inclusion of $Z$ into $X$. Define a functor $i^\diamond : \mathfrak{Sh}_X \to \mathfrak{Sh}_Z$ by
$$i^\diamond \mathcal{F} = i^{-1}(\ker(\mathcal{F} \to j_* j^{-1}\mathcal{F}))$$
where $\mathcal{F} \to j_* j^{-1}\mathcal{F}$ is the obvious morphism. Show that $(i_*, i^\diamond)$ is an adjoint pair. Also show that $i^\diamond$ is left-exact, and that $i^! = Ri^\diamond$.

6. Let $i : \{x\} \hookrightarrow X$ be the inclusion of a point. Show that $i^*\mathbb{D}\mathcal{F}^\bullet \simeq \mathbb{D}i^!\mathcal{F}^\bullet$. Do not use the more general version of this fact that was proved in class—the proof of that statement requires this one to be known first. (*Hint*: First, using adjointness of of $Ra_!$ and $a^!$ where $a : X \to \{\mathrm{pt}\}$ is a map to a one-point space, show that
$$R\Gamma(U, \mathbb{D}\mathcal{F}^\bullet) \simeq R\operatorname{Hom}(R\Gamma_c(U, \mathcal{F}^\bullet), \mathbb{C}).$$
Then, show that
$$\varprojlim_{U \ni x} R\Gamma_c(U, \mathcal{F}^\bullet) \simeq Ri^\diamond \mathcal{F}^\bullet.$$
Finally, use the fact that
$$i^*\mathbb{D}\mathcal{F}^\bullet = \varinjlim_{U \ni x} R\Gamma(U, \mathbb{D}\mathcal{F}^\bullet)$$
to complete the proof.)

7. Give an example showing that $f^!$ is not, in a general, a derived functor. (*Hint*: if $f^!$ were a derived functor, it would have to be the derived functor of $H^0 \circ f^!$. Find an example of a map $f : X \to Y$ such that $H^0(f^!\mathcal{F})$ is always 0, but $f^!\mathcal{F}$ is not.)

8. Prove all the statements from the following list that were not proved in class.
$$R\mathcal{H}om(\mathcal{F}^\bullet, \mathbb{D}\mathcal{G}^\bullet) \simeq R\mathcal{H}om(\mathcal{G}^\bullet, \mathbb{D}\mathcal{F}^\bullet) \qquad Rf_!\mathbb{D}\mathcal{F}^\bullet \simeq \mathbb{D}Rf_*\mathcal{F}^\bullet$$
$$R\mathcal{H}om(\mathcal{F}^\bullet, \mathbb{D}\mathcal{G}^\bullet) \simeq \mathbb{D}(\mathcal{F}^\bullet \overset{L}{\otimes} \mathcal{G}^\bullet) \qquad Rf_*\mathbb{D}\mathcal{F}^\bullet \simeq \mathbb{D}Rf_!\mathcal{F}^\bullet$$