Applications of Homological Algebra — Introduction to Perverse Sheaves
Spring 2007 — P. Achar

# Problem Set 8

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The goal of this problem set is to illustrate that the concept of $t$-structure is not trivial: the heart of a nontrivial $t$-structure on the derived category of an abelian category need not be equivalent to the original abelian category.

**NOTE:** Please do not hand in Problem 1 for credit. (You may of course hand it in if you just want me to look over your solution.)

1. Let $\mathcal{A}$ (resp. $\mathcal{B}$) be the category whose objects $V$ are diagrams of complex vector spaces of the form

$$V_1 \to V_2 \leftarrow V_3 \qquad \text{resp.} \qquad V_1 \leftarrow V_2 \to V_3,$$

and in which a morphism $f : V \to W$ is a commutative diagram

$$\begin{tikzcd} V_1 \arrow[r] \arrow[d, "f_1"'] & V_2 \arrow[d, "f_2"'] & V_3 \arrow[l] \arrow[d, "f_3"'] \\ W_1 \arrow[r] & W_2 & W_3 \arrow[l] \end{tikzcd} \qquad \text{resp.} \qquad \begin{tikzcd} V_1 \arrow[d, "f_1"'] & V_2 \arrow[l] \arrow[r] \arrow[d, "f_2"'] & V_3 \arrow[d, "f_3"'] \\ W_1 & W_2 \arrow[l] \arrow[r] & W_3 \end{tikzcd}$$

Show that up to isomorphism, there are exactly three simple objects of $\mathcal{A}$ (resp. $\mathcal{B}$) are

$$\begin{aligned} E_1 &= (\mathbb{C} \to 0 \leftarrow 0) \\ E_2 &= (0 \to \mathbb{C} \leftarrow 0) \\ E_3 &= (0 \to 0 \leftarrow \mathbb{C}) \end{aligned} \qquad \text{resp.} \qquad \begin{aligned} F_1 &= (\mathbb{C} \leftarrow 0 \to 0) \\ F_2 &= (0 \leftarrow \mathbb{C} \to 0) \\ F_3 &= (0 \leftarrow 0 \to \mathbb{C}) \end{aligned}$$

2. Compute $\operatorname{Ext}^1(E_i, E_j)$ and $\operatorname{Ext}^1(F_i, F_j)$. Specifically, show that

$$\dim \operatorname{Ext}^1(E_i, E_j) = \begin{cases} 1 & \text{if there is an arrow from } V_i \text{ to } V_j \text{ in objects of } \mathcal{A}, \\ 0 & \text{otherwise.} \end{cases},$$

and similarly for $\mathcal{B}$. (*Hint*: Use the relationship between $\operatorname{Ext}^1$ and extensions given by Problem 5 of Problem Set 6.) Then, deduce that $\mathcal{A}$ and $\mathcal{B}$ are *not* equivalent categories.

3. In this problem, you will show that although $\mathcal{A}$ and $\mathcal{B}$ are inequivalent abelian categories, their derived categories are equivalent. Let $S : D(\mathcal{A}) \to D(\mathcal{B})$ be the functor that takes a complex $V^\bullet$ (whose $i$th term is the diagram $V^i = (V_1^i \xrightarrow{f_i} V_2^i \xleftarrow{g_i} V_3^i)$) to the complex whose terms are

$$S(V^\bullet)^i = (\ V_1^i \xleftarrow{(1\ 0\ 0)} V_1^i \oplus V_2^{i-1} \oplus V_3^i \xrightarrow{(0\ 0\ 1)} V_3^i\ )$$

and whose differentials $d^i_{S(V^\bullet)} : S(V^\bullet)^i \to S(V^\bullet)^{i+1}$ are given by

$$\begin{tikzcd}[column sep=large] V_1^i \arrow[d, "d_1^i"'] & V_1^i \oplus V_2^{i-1} \oplus V_3^i \arrow[l, "(1\ 0\ 0)"'] \arrow[r, "(0\ 0\ 1)"] \arrow[d, "\begin{pmatrix} d_1^i & & \\ f_i & -d_2^{i-1} & g_i \\ & & d_3^i \end{pmatrix}"'] & V_3^i \arrow[d, "d_3^i"'] \\ V_1^{i+1} & V_1^{i+1} \oplus V_2^i \oplus V_3^{i+1} \arrow[l, "(1\ 0\ 0)"] \arrow[r, "(0\ 0\ 1)"'] & V_3^{i+1} \end{tikzcd}$$

On the other hand, given a complex $W^\bullet$ in $D(\mathcal{B})$, with terms $W^i = (W_1^i \xleftarrow{h_i} W_2^i \xrightarrow{k_i} W_3^i)$, let $T(W^\bullet)$ be the complex in $D(\mathcal{A})$ given by

$$T(W^\bullet)^i = (\ W_1^i \xrightarrow{\left(\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right)} W_1^i \oplus W_2^{i+1} \oplus W_3^i \xleftarrow{\left(\begin{smallmatrix}0\\0\\1\end{smallmatrix}\right)} W_3^i\ )$$

with differentials

$$\begin{tikzcd}
W_1^i \arrow[r, "\left(\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right)"] \arrow[d, "d_1^i"'] & W_1^i \oplus W_2^{i+1} \oplus W_3^i \arrow[d, "\left(\begin{smallmatrix}d_1^i & h_i & \\ & -d_2^{i+1} & \\ & k_i & d_3^i\end{smallmatrix}\right)"'] & W_3^i \arrow[l, "\left(\begin{smallmatrix}0\\0\\1\end{smallmatrix}\right)"'] \arrow[d, "d_3^i"'] \\
W_1^{i+1} \arrow[r, "\left(\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right)"'] & W_1^{i+1} \oplus W_2^{i+2} \oplus W_3^{i+1} & W_3^{i+1} \arrow[l, "\left(\begin{smallmatrix}0\\0\\1\end{smallmatrix}\right)"]
\end{tikzcd}$$

Show that $S$ and $T$ are equivalences of categories, inverse to one another. (Essentially, you must show that $T(S(V^\bullet))$ is quasi-ismorphic to $V^\bullet$, and that $S(T(W^\bullet))$ is quasi-isomorphic to $W^\bullet$.)

4. By following the standard $t$-structure on $D(\mathcal{B})$ through $T$ over to $D(\mathcal{A})$, we obtain a nonstandard $t$-structure on $D(\mathcal{A})$. (It must be nonstandard because its heart is a copy of $\mathcal{B}$, not of $\mathcal{A}$.) Describe this $t$-structure explicitly.