Problem Set 9

March 22, 2007

In this problem set, you will show that the derived category of the heart of a t-structure on a triangulated category need not be equivalent to the original triangulated category.

Let X be the 2-sphere. Let $\mathcal{C}$ be the full subcategory of $D^b(\mathfrak{Sh}_X)$ consisting of complexes of sheaves $\mathcal{F}$ all of whose cohomology sheaves $H^i(\mathcal{F})$ are constant ordinary sheaves on X.

1. Show that $\mathcal{C}$ inherits from $D^b(\mathfrak{Sh}_X)$ the structure of a triangulated category. (The main thing to show is that any morphism can be completed to a distinguished triangle: if $\mathcal{F} \to \mathcal{G}$ is a morphism in $\mathcal{C}$, then of course there is a distinguished triangle $\mathcal{F} \to \mathcal{G} \to \mathcal{H} \to \mathcal{F}[1]$ in $D^b(\mathfrak{Sh}_X)$, but is $\mathcal{H}$ necessarily in $\mathcal{C}$?)

2. Let $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ be the t-structure obtained by restricting the standard t-structure on $D^b(\mathfrak{Sh}_X)$ to $\mathcal{C}$:

$$\begin{aligned} \mathcal{C}^{\leq 0} &= \{\mathcal{F} \mid H^i(\mathcal{F}) = 0 \text{ for all } i > 0\}, \\ \mathcal{C}^{\geq 0} &= \{\mathcal{F} \mid H^i(\mathcal{F}) = 0 \text{ for all } i < 0\}. \end{aligned}$$

   Let $\mathcal{T}$ be the heart of this t-structure. Show that $\mathcal{T}$ is equivalent to the category $\mathfrak{Vect}_{\mathbb{C}}$ of finite-dimensional complex vector spaces.

3. The bounded derived category $D^b(\mathcal{T})$ is also a triangulated category equipped with a t-structure (the standard one). A natural question is: is there an equivalence of categories $\Phi : D^b(\mathcal{T}) \to \mathcal{C}$ that respects all the additional structures on both sides? Such a functor should:

   - be compatible with the shift functors in both categories,
   - take distinguished triangles to distinguished triangles,
   - take $D^b(\mathcal{T})^{\leq 0}$ to $\mathcal{C}^{\leq 0}$ and $D^b(\mathcal{T})^{\geq 0}$ to $\mathcal{C}^{\geq 0}$, and
   - be the identity functor on $\mathcal{T}$.

   Show that there is no such functor. (*Hint*: Consider the constant sheaf $\underline{\mathbb{C}}_X$, which can be regarded as an object of $\mathcal{C}$, of $\mathcal{T}$, or of $D^b(\mathcal{T})$. Show that

$$\mathrm{Hom}_{\mathcal{C}}(\underline{\mathbb{C}}_X, \underline{\mathbb{C}}_X[2]) \simeq \mathbb{C} \qquad \text{but} \qquad \mathrm{Hom}_{D^b(\mathcal{T})}(\underline{\mathbb{C}}_X, \underline{\mathbb{C}}_X[2]) = 0.$$

   To calculate the Hom group in $\mathcal{C}$, you will need to use the following fact about the 2-sphere:

$$H^i(X, \mathbb{C}) \simeq \begin{cases} \mathbb{C} & \text{if } i = 0 \text{ or } 2, \\ 0 & \text{otherwise.} \end{cases}$$

   You may use this fact without proof. On the other hand, calculating a Hom group in $D^b(\mathcal{T})$ is, by the previous exercise, the same as calculating a Hom group in $D^b(\mathfrak{Vect}_{\mathbb{C}})$.)