Algebraic Groups Quick Reference

Preliminary Version

G denotes a connected algebraic group throughout. Numbers in parentheses indicate sections in Humphreys' book.

Elements of algebraic groups

1. **Jordan decomposition:** Every element $x \in G$ can be written in a unique way as $x = su$, where s is semisimple, u is unipotent, and s and u commute. (15.3)
2. The set G_u of unipotent elements is closed. If G is solvable, G_u is a normal subgroup. (Usually, the set G_s of semisimple elements is not closed and not a subgroup.) (15.3, 19.1)
3. Every semisimple element is contained in a maximal torus. (19.3; 20.2)
4. Every element is contained in a Borel subgroup. (20.2)

Structure theorems

5. Any group G has a unique maximal normal connected solvable subgroup $R(G)$, called the **radical**, and a unique maximal normal connected unipotent subgroup $R_u(G)$, called the **unipotent radical**. (19.5)
6. A connected group consisting of semisimple elements is a torus. (16.2)
7. A unipotent group is isomorphic to a subgroup of $U(n, K)$. (17.5)
8. **Lie–Kolchin theorem:** A solvable group is isomorphic to a subgroup of $T(n, K)$. (17.6)
9. If G is nilpotent, then G_s is a closed, connected subgroup (hence a torus), and $G \simeq G_s \times G_u$. (19.2)
10. If G is solvable, then $G \simeq T \ltimes G_u$, where T is some maximal torus. (19.3)

Group actions

11. For any closed subgroup $H \subset G$, there is a representation $G \to GL(V)$ and a line $V_1 \subset V$ such that $H = \{x \in G \mid x(L) \subset L\}$. (11.2)
12. An orbit of minimal dimension is closed. (8.3)
13. Conjugacy classes of semisimple elements are closed. More generally, if $s \in G$ is semisimple and normalizes $H \subset G$, then $Cl_H(s)$ is closed. (18.2)
14. If G is a solvable group acting on a projective variety X, then G has a fixed point on X. (21.2)

Conjugacy theorems

15. All maximal tori are conjugate to one another. (19.3; 21.3)
16. All Borel subgroups are conjugate to one another. (21.3)

Centralizers

17. If $U \subset G$ is a connected unipotent group, and $s \in G$ is semisimple, then $C_U(s)$ is connected. (18.3)
18. If $S \subset G$ is a torus and $H \subset G$ is a closed subgroup normalized by S, there is an element $s \in S$ such that $C_H(S) = C_H(s)$. In fact, such s form an open dense subset in S. (16.4)
19. If $S \subset G$ is a torus, $C_G(S)$ is connected. If G is reductive, then $C_G(S)$ is also reductive. (19.4; 22.3; 26.2)
20. If $T \subset G$ is a maximal torus, then $C_G(T)$ is nilpotent. If G is reductive, then $C_G(T) = T$. (21.4; 26.2)
21. If $B \subset G$ is a Borel subgroup, then $Z(B) = Z(G)$. (21.4; 22.2)

Normalizers

22. If $S \subset G$ is a torus, then $N_G(S)^\circ = C_G(S)$. If G is solvable, $N_G(S) = C_G(S)$. (21.4; 19.4)
23. If $B \subset G$ is a Borel subgroup, then $N_G(B) = B$. Therefore, the set of all Borel subgroups is in bijection with G/B. (23.1)