Math 7290 Fall 2010

Introduction to Lie Theory P. Achar

Problem Set 1

Due: September 9, 2010

1. Show that $GL(n,\mathbb{C})$ is an algebraic group.

2. Let $G = \{z \in \mathbb{C} \mid |z| = 1\}$. G is a Lie group under multiplication; it is homeomorphic to a circle. Show that G is *not* a complex algebraic group. (It is a "real algebraic group," however.)

3. Let $R_1 \subset V_1$ and $R_2 \subset V_2$ be root systems. Let $V = V_1 \oplus V_2$, and let $R = R_1 \cup R_2 \subset V$. Let $V' \subset V$ be a subspace, and let $T \subset R \cap V'$ be a subset. Show that if T is itself a root system, then the sets
$$T_1 = T \cap R_1 \subset V_1 \qquad \text{and} \qquad T_2 = T \cap R_2 \subset V_2$$
are root systems as well, and that $T = T_1 \cup T_2$.

4. Classify the irreducible *nonreduced* root systems. These are related to "super Lie algebras."
   (a) Show that for any root system R, if $\alpha \in R$, then $-\alpha \in R$.
   (b) Recall that the only other possible multiples of α that may belong to R are $\pm\frac{1}{2}\alpha$ and $\pm 2\alpha$. Show that $\pm\frac{1}{2}\alpha$ and $\pm 2\alpha$ cannot both belong to R.

   Combining these two steps, we see that there are exactly three kinds of roots:
   - A root α is *reduced* if $\frac{1}{2}\alpha \notin R$ and $2\alpha \notin R$.
   - A root α is *short nonreduced* if $\frac{1}{2}\alpha \notin R$ but $2\alpha \in R$.
   - A root α is *long nonreduced* if $\frac{1}{2}\alpha \in R$ but $2\alpha \notin R$.

   Of course, 2(a short nonreduced root) is a long nonreduced root, and vice versa.
   (c) Let R be an irreducible nonreduced root system. Let $R_s = \{\text{reduced roots}\} \cup \{\text{short nonreduced roots}\}$. Show that R_s is an irreducible reduced root system. Same for $R_l = \{\text{reduced roots}\} \cup \{\text{long nonreduced roots}\}$.
   (d) Show that a base for the root system R_s must contain at least one short nonreduced root. (*Hint:* If not, show that that base would also be a base for R_l. Deduce a contradiction.)
   (e) What can you say about the relationship of the Dynkin diagram of R_s to that of R_l?
   (f) Find all pairs of connected Dynkin diagrams that are related in the way you described above. To complete the classification, for each such pair, write down explicitly the vectors in the nonreduced root system R.

5. In this problem, let V be a *complex* vector space. A subset $R \subset V$ is called a *reduced Gaussian root system* if it satisfies the following four axioms:
   (R1) R is finite and spans V, and $0 \notin R$.
   (R2) For all $\alpha \in R$, there is an element $\alpha^\vee \in V^*$ such that $\alpha^\vee(\alpha) = 1 - i$, and such that $s_{\alpha,\alpha^\vee}(R) = R$. Here, $s_{\alpha,\alpha^\vee} : V \to V$ is the linear map defined as usual by $x \mapsto x - \alpha^\vee(x)\alpha$.
   (R3) For all $\alpha, \beta \in R$, we have $\alpha^\vee(\beta) \in \mathbb{Z}[i]$.
   (R4) For all $\alpha \in R$, we have $\mathbb{C}\alpha \cap R = \{\pm\alpha, \pm i\alpha\}$.

   Develop a full theory of reduced Gaussian root systems, and classify the irreducible ones. (*Note*: This is quite a long problem, and may not be feasible to do in full. Begin by making a list of steps to carry out, and carry out as many of them as you can. I think there are only two irreducible Gaussian root systems, of ranks 1 and 2.)