Problem Set 2

Due: September 28, 2010

1. Classify the 2-dimensional Lie algebras up to isomorphism. In particular, show that every 2-dimensional Lie algebra is solvable.

2. Show that $\mathfrak{sl}_2$ is a simple Lie algebra. (*Hint*: Note that $\dim \mathfrak{sl}_2 = 3$. First show, using the previous exercise, that if $\mathfrak{sl}_2$ had a nontrivial ideal, then $\mathfrak{sl}_2$ itself would be solvable. Then, show directly by computing $\mathcal{D}(\mathfrak{sl}_2)$ that $\mathfrak{sl}_2$ is not solvable.)

3. The notion of a Lie algebra makes sense over an arbitrary field, not just over $\mathbb{C}$, but some of the basic theory doesn't go through. Give a counterexample to Lie's theorem over a field of characteristic 2. (*Hint*: if $\Bbbk$ is an algebraically closed field of characteristic 2, show that $\mathfrak{sl}_2(\Bbbk)$ is solvable, but that there is no flag in $\Bbbk^2$ that is preserved by $\mathfrak{sl}_2(\Bbbk)$.)

4. Let $\mathfrak{g} \subset \mathfrak{gl}(V)$ be a nilpotent Lie subalgebra. Is it true that every element of $\mathfrak{g}$ is necessarily a nilpotent endomorphism of V? Give a proof or a counterexample.