Problem Set 3

Due: October 19, 2010

1. Prove that the set of diagonal matrices in $\mathfrak{sl}_n$ is a Cartan subalgebra. (We used this example in class many times, but we never actually proved this statement.)

2. Let X be an affine variety in $\mathbb{C}^N$. Prove that X has only finitely many connected components. (*Hint*: Show that for any component $X_1 \subset X$, there is a polynomial $f \in \mathbb{C}[x_1, \ldots, x_N]$ such that $f|_{X \smallsetminus X_1} = 0$ but $f|_{X_1} \neq 0$. Then use the fact that $\mathbb{C}[x_1, \ldots, x_N]$ is a noetherian ring.)

3. Let $X \subset \mathbb{C}^N$ be a variety defined by equations
$$f_1(x_1, \ldots, x_N) = f_2(x_1, \ldots, x_N) = \cdots = f_k(x_1, \ldots, x_N) = 0.$$
Assume that this set of equations is *reduced*, meaning that every polynomial $p \in \mathbb{C}[x_1, \ldots, x_N]$ that vanishes on all points of X is in the ideal generated by $f_1, \ldots, f_k$. Consider a point $a = (a_1, \ldots, a_N) \in X$. Show that the tangent space $T_z X$ can be identified with the following subspace of $\mathbb{C}^N$:
$$\left\{ (b_1, \ldots, b_N) \,\middle|\, \text{for all } j,\ \sum_{i=1}^{N} b_i \frac{\partial f_j}{\partial x_i}(a_1, \ldots, a_N) = 0 \right\}.$$

4. Now, apply the previous problem to describe the tangent spaces at the identity element of various algebraic groups. You may assume without proof that the sets of defining equations given below are reduced.
   (a) For $SL_n \subset \mathbb{C}^{n^2}$, defined by the equation $\det(A) - 1 = 0$, show that
$$T_e SL_n \cong \{A \in \mathbb{C}^{n^2} \mid \operatorname{tr} A = 0\}.$$
   (b) Fix an invertible matrix $B \in \mathbb{C}^{n^2}$ that is either symmetric or skew-symmetric, and let $G = \{A \in \mathbb{C}^{n^2} \mid {}^t\!ABA = B\}$, where ${}^t\!A$ denotes the transpose of A. (Note that the matrix equation ${}^t\!ABA = B$ is a set of n^2 polynomial equations.) Show that
$$T_e G \cong \{A \in \mathbb{C}^{n^2} \mid {}^t\!AB + BA = 0\}.$$