Problem Set 4

Due: November 18, 2010

1. Prove that every connected abelian unipotent group is isomorphic to a product of copies of $\mathbb{G}_\mathrm{a}$. (This statement is not true in positive characteristic, so you should try to point out where in your proof you use the fact that $\mathbb{C}$ has characteristic 0.)

2. Let G and H be two algebraic groups. Show that $\mathfrak{Lie}(G \times H) \cong \mathfrak{Lie}(G) \times \mathfrak{Lie}(H)$.

3. The groups $\mathbb{G}_\mathrm{m} \times \mathbb{G}_\mathrm{m}$ and $\mathbb{G}_\mathrm{a} \times \mathbb{G}_\mathrm{a}$ have isomorphic Lie algebras: namely, both are 2-dimensional abelian Lie algebras.

   (a) Show that every subalgebra of $\mathfrak{Lie}(\mathbb{G}_\mathrm{a} \times \mathbb{G}_\mathrm{a})$ is algebraic.

   (b) Classify the algebraic Lie subalgebras of $\mathfrak{Lie}(\mathbb{G}_\mathrm{m} \times \mathbb{G}_\mathrm{m})$.

   This problem shows that the notion of "algebraic Lie subalgebra" is not, in general, intrinsic to the Lie algebra, but depends on the algebraic group as well. However, the situation is better for semisimple Lie algebras.

4. The previous problem shows yields examples where $\dim \mathcal{A}(\mathfrak{h}) > \dim \mathfrak{h}$.

   (a) Give examples showing that the difference $\dim \mathcal{A}(\mathfrak{h}) - \dim \mathfrak{h}$ can be made arbitrarily large.

   (b) Recall that if $\dim \mathfrak{h} = 1$, then $\mathcal{A}(\mathfrak{h})$ is a connected abelian group, so we have $\mathcal{A}(\mathfrak{h}) \cong T \times \mathbb{G}_\mathrm{a}^n$, where T is a torus, and U is a connected abelian unipotent group. Can every connected abelian group occur as $\mathcal{A}(\mathfrak{h})$ for some 1-dimensional Lie subalgebra $\mathfrak{h}$? (*Hint*: Think about Problem 3(a).)

5. Let G be an algebraic group.

   (a) Give necessary and sufficient conditions on $\mathbf{A}(G)$ and $\mu : \mathbf{A}(G) \to \mathbf{A}(G) \otimes \mathbf{A}(G)$ for G to be abelian.

   (b) Show directly (without resorting to the results of Chapter 24) that if the criterion from part (a) holds, then any two left-invariant derivations $X, Y : \mathbf{A}(G) \to \mathbf{A}(G)$ commute.

   (c) Give a counterexample to the converse statement: find G such that any two left-invariant derivations commute, but such that the criterion of part (a) does not hold. (Why does this not contradict the results of Chapter 24?)