Math 7290 — Fall 2010

Introduction to Lie Theory — P. Achar

Problem Set 5

(Optional)

1. Prove *Schur's Lemma*, which states that if (V, π) is an irreducible finite-dimensional representation of a group G or a Lie algebra $\mathfrak{g}$, then the only G- or $\mathfrak{g}$-equivariant maps $V \to V$ are given by multiplication by a scalar. What happens for infinite-dimensional representations?

2. Prove the Poincaré–Birkhoff–Witt theorem for $\mathfrak{sl}_2$.

3. Let $\mathfrak{h} \subset \mathfrak{sl}_2$ denote the usual Cartan subalgebra consisting of diagonal matrices. Let us identify the dual space $\mathfrak{h}^*$ with $\mathbb{C}$ by sending $\chi \in \mathfrak{h}^*$ to the complex number $\chi(\left[\begin{smallmatrix} 1 & \\ & -1 \end{smallmatrix}\right])$. Under this identification, a weight $\chi \in \mathfrak{h}^*$ is integral (resp. dominant integral) if and only if the corresponding complex number is an integer (resp. a nonnegative integer).

   Prove that if χ is *not* dominant integral, then the Verma module $M(\chi)$ is irreducible. What weights occur in $M(\chi)$? What are the dimensions of the corresponding weight spaces?

4. Show that the "Casimir element" $\Omega = \frac{1}{2}H^2 + EF + FE$ is in the center of the ring $U(\mathfrak{sl}_2)$. Explain why Schur's Lemma implies that Ω acts on finite-dimensional irreducible representations by a scalar, and then determine the scalar by which it acts on each one. How does Ω act on Verma modules?

5. Let $G = SL_2$, and let $T \subset G$ denote the usual maximal torus consisting of diagonal matrices. Identify $X^*(T)$ with $\mathbb{Z}$ by the correspondence $(\left(\begin{smallmatrix} t & \\ & t^{-1} \end{smallmatrix}\right) \mapsto t^n) \longleftrightarrow n$. Let $B \subset G$ be the usual Borel subgroup consisting of upper-triangular matrices. If $\chi \in X^*(T)$ corresponds to $n \in \mathbb{Z}$, show that there is a G-equivariant isomorphism

   $$\Gamma(G/B, G \times^B \mathbb{C}_\chi) \simeq \{f : \mathbb{C}^2 \to \mathbb{C} \mid f(cx, cy) = c^n f(x, y)\},$$

   where G acts on the latter space by $(\mathfrak{g} \cdot f)(x, y) = f(g^{-1} \cdot (x, y))$. (On the right-hand side, $G = SL_2$ acts on $\mathbb{C}^2$ by the obvious action.)