Math 2020 Spring 2011
Solving Discrete Problems P. Achar

Homework Set 11

Due: April 25, 2011

1. Proposition D.1.
2. Proposition D.2. (*Warning*: The proof of this is *not* similar to those of Propositions 1.7 and 1.8 from Chapter 1: those proofs used the commutative axioms for $\mathbb{Z}$, but here, you need to give a proof that's valid even for nonabelian groups.)
3. Proposition D.14.

For the next few problems, you will need the following definition. Let G and H be two groups. They are said to be **isomorphic** if there is a bijection $f : G \to H$ such that for all $g_1, g_2 \in G$, we have

$$f(g_1 \cdot g_2) = f(g_1) \cdot f(g_2). \tag{$*$}$$

In this case, f is said to be an **isomorphism**. Note that "$\cdot$" means *different* things on the two sides of this equation: on the left, it's the operation in G, but on the right, it's the operation in H.

Examples: Consider the groups $S_2 = \{\begin{smallmatrix}1&2\\1&2\end{smallmatrix}, \begin{smallmatrix}1&2\\2&1\end{smallmatrix}\}$ and $H = \{\begin{smallmatrix}1&2&3\\1&2&3\end{smallmatrix}, \begin{smallmatrix}1&2&3\\2&1&3\end{smallmatrix}\}$. (We discussed H in class on Wednesday, April 13; it's an abelian subgroup of S_3.) The function $f : S_2 \to H$ given by

$$f(\begin{smallmatrix}1&2\\1&2\end{smallmatrix}) = \begin{smallmatrix}1&2&3\\1&2&3\end{smallmatrix}, \qquad f(\begin{smallmatrix}1&2\\2&1\end{smallmatrix}) = \begin{smallmatrix}1&2&3\\2&1&3\end{smallmatrix}$$

is an isomorphism. (Check that $(*)$ holds for all $g_1, g_2 \in S_2$.) So S_2 and H are isomorphic. That still doesn't mean that *every* bijection between them is an isomorphism: the function $q : S_2 \to H$ given by

$$q(\begin{smallmatrix}1&2\\1&2\end{smallmatrix}) = \begin{smallmatrix}1&2&3\\2&1&3\end{smallmatrix}, \qquad q(\begin{smallmatrix}1&2\\2&1\end{smallmatrix}) = \begin{smallmatrix}1&2&3\\1&2&3\end{smallmatrix}$$

is a bijection but is *not* an isomorphism, because $q((\begin{smallmatrix}1&2\\2&1\end{smallmatrix}) \cdot (\begin{smallmatrix}1&2\\2&1\end{smallmatrix})) \neq q(\begin{smallmatrix}1&2\\2&1\end{smallmatrix}) \cdot q(\begin{smallmatrix}1&2\\2&1\end{smallmatrix})$. (Check that.)

4. Prove that for any isomorphism $f : G \to H$, we have $f(1_G) = 1_H$. Here, 1_G denotes the identity element of G, and 1_H denotes the identity element of H. (*Hint*: You will need to use the "uniqueness" part of Proposition D.1.)
5. For any group G, the identity function $\mathrm{id}_G : G \to G$ is an isomorphism from G to itself. Find an example of an isomorphism $f : \mathbb{Z}_4 \to \mathbb{Z}_4$ *other than* the identity function.
6. Prove that S_3 is *not* isomorphic to $\mathbb{Z}_6$. (They both have the same number of elements—namely, six—so there certainly exist bijections from S_3 to $\mathbb{Z}_6$. You have to prove that no bijection satisfies the extra condition $(*)$.)
7. (Bonus) Cayley's Theorem: Every finite group is isomorphic to a subgroup of a symmetric group.