Homework Set 12

Due: May 2, 2011

1. Let G be the group of symmetries of an equilateral triangle. Prove that G is isomorphic to S_3.

2. *Either:* Proposition D.19.

   Or: Let G be the group of all rational numbers, with $+$ as the operation. Prove that G is not finitely generated. (For either problem, feel free to use common facts that you know about rational numbers. You will need to think about what denominators you can get by "multiplying" (which means $\cdot$ in the first problem, and $+$ in the second) fractions from a finite list.)

3. Proposition D.13. In this question, some work is required to get a handle on the notion of "even permutations." The hardest part is proving that the product of two even permutations is even. Here is a suggested outline of how to do this. (This is not the only way to do it; see the remarks after part (e).)

   (a) Let P_n be the set of all pairs of numbers $\{i, j\}$ such that $1 \le i \le n$ and $1 \le j \le n$ but $i \ne j$. For example,
   $$P_4 = \{\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}\}.$$
   Given $\sigma \in S_n$, define a function $h_\sigma : P_n \to P_n$ by the formula
   $$h_\sigma(\{i,j\}) = \{\sigma(i), \sigma(j)\}.$$
   Prove that h_σ is a bijection.

   (b) Given a permutation $\sigma \in S_n$, define a function $f_\sigma : P_n \to \mathbb{Z}_2$ by
   $$f_\sigma(\{i,j\}) = \begin{cases} [0] & \text{if } i < j \text{ and } \sigma(i) < \sigma(j), \\ [1] & \text{if } i < j \text{ and } \sigma(i) > \sigma(j). \end{cases}$$
   Next, let
   $$N_\sigma = \text{the number of elements } \{i,j\} \in P_n \text{ such that } f_\sigma(\{i,j\}) = [1].$$
   Convince yourself that
   $$\sigma \text{ is even} \qquad \text{if and only if} \qquad N_\sigma \text{ is even.}$$
   (There is nothing to prove here; this is just a way of rewriting the definition of "even permutation.")

   (c) Prove that $f_{\tau\sigma}(\{i,j\}) = f_\tau(h_\sigma(\{i,j\})) \oplus f_\sigma(\{i,j\})$.

   (d) By definition, $N_\tau =$ the number of elements $\{i,j\} \in P_n$ such that $f_\tau(\{i,j\}) = [1]$. Using part (a), prove that
   $$N_\tau = \text{the number of elements } \{i,j\} \in P_n \text{ such that } f_\tau(h_\sigma(\{i,j\})) = [1].$$

   (e) Let us define
   $$M_{\tau,\sigma} = \text{the number of elements } \{i,j\} \in P_n \text{ such that } f_\tau(h_\sigma(\{i,j\})) = [1] \textit{ and } f_\sigma(\{i,j\}) = [1].$$

In order to have $f_{\tau\sigma}(\{i,j\}) = [1]$, note that one of $f_\tau(h_\sigma(\{i,j\}))$ and $f_\sigma(\{i,j\})$ must be equal to $[1]$, and the other one must be $[0]$. Use this observation to show that

$$N_{\tau\sigma} = N_\tau + N_\sigma - 2M_{\tau,\sigma}.$$

Then, deduce that the product of two even permutations is even, and that the product of an even permutation and an odd permutation is odd.

Alternate approach: Here is an outline of a totally different approach that could replace steps (a)–(e) above. If you know some linear algebra, you might prefer this to what I outlined above. Let

$$T_n = \{n \times n \text{ matrices with exactly one "1" in each row and each column, and "0"'s elsewhere}\}.$$

First, prove that T_n is a subgroup of $GL_n(\mathbb{R})$. (That is, it's a group under matrix multiplication.) Let $b : S_n \to T_n$ be the function given by the following rule:

$$b_n(\sigma) = \text{the matrix whose } i\text{th column has a 1 in row } \sigma(i)\text{, and 0's elsewhere.}$$

Show that b is an isomorphism. Finally, show that $\sigma \in S_n$ is even if and only if $\det b(\sigma) = 1$. (Odd permutations have $\det b(\sigma) = -1$.) Then, deduce that the product of two even permutations is even, and that the product of an even permutation and an odd permutation is odd.

The rest of the proof is the same, regardless of which approach you took.

(f) Prove that the inverse of an even permutation is even. This step completes the proof that the even permutations form a subgroup.

(g) Write down a specific *odd* permutation ρ_1. Recall that $A_n \subseteq S_n$ denotes the set of even permutations, so $S_n - A_n$ is the set of odd permutations. Consider the function

$$q : A_n \to S_n - A_n \qquad \text{given by} \qquad q(\sigma) = \rho_1\sigma.$$

Prove that q is a bijection. Deduce that the number of even permutations in S_n is equal to the number of odd permutations, and then that A_n contains $n!/2$ elements.