Notes on Chapter 3

Universal Quantifier. A sentence with a universal quantifier can often be rewritten using "Let" or "If ... then." The four sentences below are all different ways of expressing the same thing. Note that in the fourth example, the quantified variable (the subset of $\mathbb{N}$) isn't given a name.

$\forall A \subseteq \mathbb{N}$, if A is nonempty, then A has a smallest element.
Let $A \subseteq \mathbb{N}$. If is a nonempty, then A has a smallest element.
If $A \subseteq \mathbb{N}$ is nonempty, then A has a smallest element.
Every nonempty subset of $\mathbb{N}$ has a smallest element.

Universal quantifiers in proofs. Remember that:

- when *proving* a statement involving $\forall x$, you're not allowed to impose any extra conditions or restrictions on x that aren't already in the statement. (But you can break up the proof into parts where you consider different restrictions on x, as long as they cover all possible cases when taken together.)
- when *using* a statement involving $\forall x$ in the proof of something else, you get to pick a specific value for x.

Existential Quantifier. Here are four more sentences that all have the same meaning as each other:

$\exists x \in \mathbb{Z}$ such that $x^2 - 1 = 0$.
There is an integer x such that $x^2 - 1 = 0$.
For some $x \in \mathbb{Z}$, $x^2 - 1 = 0$.
The equation $x^2 - 1 = 0$ has a solution.

Important note: a "$\exists$ ... such that ..." sentence just says that there is *at least one* element (or set, or whatever) with the required property; it doesn't say anything about whether there are more. You can add the word "unique" if you mean that there is exactly one. The following two sentences have the same meaning:

$\exists! \, x \in \mathbb{Z}$ such that $x^2 - 1 = 0$.
There is a unique integer x such that $x^2 - 1 = 0$.

Of course, these sentences are both false! What's true is this:

$\exists x, y \in \mathbb{Z}$ such that $x^2 - 1 = 0$ and $y^2 - 1 = 0$ and $x \neq y$.
There are two distinct integers $x \in \mathbb{Z}$ such that $x^2 - 1 = 0$.
The equation $x^2 - 1 = 0$ has two solutions.

Existential quantifiers in proofs. In many ways, $\exists$ behaves in the opposite way to $\forall$:

- when *proving* a statement involving $\exists x$, you just need to produce a single specific example for x that has the required property. If there's more than one way to do this, you get to pick whatever you think is the easiest.
- when *using* a statement involving $\exists x$ in the proof of something else, you're not allowed to impose any extra assumption on x beyond what's already in the statement.

Variables. Remember that you can't use a variable in a proposition or a proof unless you have already given that variable a meaning. Variables acquire a meaning when you have explicitly defined them, or if they come from a quantifier. Examples:

> Let $x = 5$. Then x is a solution to $x^2 - m = 0$. (meaningless sentence—neither true nor false)
> Let $x = 5$ and let $m = 25$. Then $x^2 - m = 0$. (true)
> Let $m \in \mathbb{Z}$. Then there is an $x \in \mathbb{Z}$ such that $x^2 - m = 0$. (meaningful, but false)

Note that the word "let" is being used in two different ways here: in the first two sentences, we're making explicit definitions, but in the last sentence, the "let" is a version of the universal quantifier.

Implications. An implication is an "If ... then ..." statement. Another way to write "If P, then Q" is "$P \Rightarrow Q$." Two other statements you can make from an implication are:

the *converse:* $Q \Rightarrow P$ and the *contrapositive:* (not Q) $\Rightarrow$ (not P).

Remember that the original implication is logically equivalent to its contrapositive. (But the converse is just a different statement; it might be true or false independently of whether the original implication is true or false.) Two ways to prove an implication $P \Rightarrow Q$ are:

- *Direct proof*: Assume that P is true. Deduce Q using P, axioms, and previously proved propositions.
- *Proof by contradiction*: Assume that P is true *and* that Q is false. Then deduce something that contradicts one of the assumptions, or an axiom, or a previously proved proposition. In particular, you might end up deducing that P is false. In that case, the "proof by contradiction" is really the same as a direct proof of the contrapositive.

Negations. Remember that negation swaps $\forall$ and $\exists$, and it swaps "and" and "or."

Original sentence	*Negation*
$\forall x$, P.	$\exists x$ such that not P.
$\exists x$ such that P.	$\forall x$, not P.
P and Q.	(Not P) or (not Q).
P or Q.	(Not P) and (not Q).
If P, then Q.	P and (not Q).

Make sure you understand from examples why these rules make sense. Here is a non-mathematical example that involves quantifiers as well as "and" and "or."

> (*) Everyone in the class was born on a weekday or on an even-numbered date.

Remember that "or" always means "one, or the other, or both." People born on Tuesday the 8th count for this statement. Let's rewrite it in a quasi-mathematical way:

> $\forall$ students S in the class, S's date of birth was on a weekday *or* was on an even date.

The negation of this sentence is

> $\exists$ student S such that S's date of birth was on a weekend *and* on an odd date.

In plain English,

> (†) Someone in the class was born on an odd-numbered date on a weekend.

I don't know which one of (*) or (†) is true. But since they are negations of each other, it must be the case that exactly one of those two sentences is true, and the other one is false. There is no way to fill a class with students that makes (*) and (†) both true, or both false. (Which one of (*) or (†) is true for an empty class?)