Problem Set 2

Due: September 19, 2012

1. Let A be a ring. Let $\mathcal{F}$ be a sheaf of A-modules, and let $\underline{A}_X$ be the constant sheaf on X with value A. Prove that
$$\mathcal{H}om(\underline{A}_X, \mathcal{F}) \cong \mathcal{F} \qquad \text{and} \qquad \underline{A}_X \otimes \mathcal{F} \cong \mathcal{F}.$$

2. Let $\mathcal{Q}$ be the "square-root sheaf" on $X = \mathbb{C} \smallsetminus \{0\}$. Show that $\mathcal{Q} \otimes \mathcal{Q} \cong \underline{\mathbb{C}}$.

3. (Hartshorne, Exercise 1.19(c)) Let $U \subset X$ be an open set, let $Z = X \smallsetminus U$ be its complement, and let $j : U \hookrightarrow X$ and $i : Z \hookrightarrow X$ be the inclusion maps. Given a sheaf $\mathcal{F}$ on X, show that there is a short exact sequence of sheaves
$$0 \to j_!(\mathcal{F}|_U) \to \mathcal{F} \to i_*(\mathcal{F}|_Z) \to 0.$$

4. Let $\mathfrak{Sh}(X)$ be the category of sheaves on X, and let $\mathfrak{PSh}(X)$ be the category of presheaves on X. Let $I : \mathfrak{Sh}(X) \to \mathfrak{PSh}(X)$ be the inclusion functor (note that $\mathfrak{Sh}(X)$ is a subcategory of $\mathfrak{PSh}(X)$). We also have the sheafification functor ${}^+ : \mathfrak{PSh}(X) \to \mathfrak{Sh}(X)$. Show that $({}^+, I)$ is an adjoint pair of functors.

5. If $\mathcal{E}$ and $\mathcal{F}$ are local systems, show that $\mathcal{H}om(\mathcal{E}, \mathcal{F})$ and $\mathcal{E} \otimes \mathcal{F}$ are as well. What are their ranks in terms of $\operatorname{rank} \mathcal{E}$ and $\operatorname{rank} \mathcal{F}$? (The **rank** of a local system is the dimension of any of its stalks.)