Problem Set 2
Due: September 19, 2012

1. Let $A$ be a ring. Let $\mathcal{F}$ be a sheaf of $A$-modules, and let $\underline{A}_X$ be the constant sheaf on $X$ with value $A$. Prove that

$$\text{Hom}(\underline{A}_X, \mathcal{F}) \cong \mathcal{F} \quad \text{and} \quad \underline{A}_X \otimes \mathcal{F} \cong \mathcal{F}.$$

2. Let $\mathcal{Q}$ be the “square-root sheaf” on $X = \mathbb{C} \setminus \{0\}$. Show that $\mathcal{Q} \otimes \mathcal{Q} \cong \mathbb{C}$.

3. (Hartshorne, Exercise 1.19(c)) Let $U \subset X$ be an open set, let $Z = X \setminus U$ be its complement, and let $j : U \hookrightarrow X$ and $i : Z \hookrightarrow X$ be the inclusion maps. Given a sheaf $\mathcal{F}$ on $X$, show that there is a short exact sequence of sheaves

$$0 \to j_!(\mathcal{F}|_U) \to \mathcal{F} \to i_*(\mathcal{F}|_Z) \to 0.$$

4. Let $\mathcal{Sh}(X)$ be the category of sheaves on $X$, and let $\mathcal{PSh}(X)$ be the category of presheaves on $X$. Let $I : \mathcal{Sh}(X) \to \mathcal{PSh}(X)$ be the inclusion functor (note that $\mathcal{Sh}(X)$ is a subcategory of $\mathcal{PSh}(X)$). We also have the sheafification functor $^+ : \mathcal{PSh}(X) \to \mathcal{Sh}(X)$. Show that $(^+, I)$ is an adjoint pair of functors.

5. If $\mathcal{E}$ and $\mathcal{F}$ are local systems, show that $\text{Hom}(\mathcal{E}, \mathcal{F})$ and $\mathcal{E} \otimes \mathcal{F}$ are as well. What are their ranks in terms of rank $\mathcal{E}$ and rank $\mathcal{F}$? (The rank of a local system is the dimension of any of its stalks.)