Problem Set 3

Due: October 15, 2012

1. In class, we proved that there are enough flabby sheaves by showing that there is a natural map $\mathcal{F} \to \mathcal{F}^\sharp$, where
$$\mathcal{F}^\sharp(U) = \{s : U \to \coprod_{x \in U} \mathcal{F}_x \mid s(x) \in \mathcal{F}_x\}.$$
Prove that in the category of sheaves of complex vector spaces, $\mathcal{F}^\sharp$ is actually an injective object, so this shows that the category has enough injectives. How would you need to modify this construction for sheaves of abelian groups? (*Hint for the second part:* The first part will rely on the fact that every complex vector space—including every stalk $\mathcal{F}_x$—is an injective object in the category of vector spaces.)

2. Prove that any injective sheaf is flabby.

3. (Uniqueness of adjoint functors) Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor of abelian or triangulated categories, and let $G, H : \mathcal{B} \to \mathcal{A}$ be two functors that are both right adjoint to F. Show that $G \simeq H$. *I.e.*, show that there is a rule η that assigns to every object $B \in \mathcal{B}$ a morphism $\eta(B) : G(B) \to H(B)$ in $\mathcal{A}$ such that for every morphism $f : B \to C$ in $\mathcal{B}$, the following square commutes:
$$\begin{tikzcd} G(B) \arrow[r, "\eta(B)"] \arrow[d, "G(f)"'] & H(B) \arrow[d, "H(f)"] \\ G(C) \arrow[r, "\eta(C)"'] & H(C) \end{tikzcd}$$

4. Let $j : U \hookrightarrow X$ be an inclusion of an open set. Show that j^* is right-adjoint to $j_!$. (In class, we gave a rather complicated construction for a right adjoint to $f_!$ in general. This problem, combined with the previous one, shows that for an open inclusion j, $j^! \cong j^*$.)

5. Let $j : U \hookrightarrow X$ be an inclusion of an open set. Let $Z = X \smallsetminus U$, and let $i : Z \hookrightarrow X$ be the inclusion of Z into X. Define a functor $i^\diamond : \mathfrak{Sh}_X \to \mathfrak{Sh}_Z$ by
$$i^\diamond \mathcal{F} = i^*(\ker(\mathcal{F} \to j_* j^* \mathcal{F}))$$
where $\mathcal{F} \to j_* j^* \mathcal{F}$ is the obvious morphism. Show that $(i_*, i^\diamond)$ is an adjoint pair. Also show that $i^\diamond$ is left-exact, and that $i^! = Ri^\diamond$.