Derived Categories Cheat Sheet

1. A **triangulated category** is an additive category $\mathscr{T}$ equipped with an automorphism (called **shift** or **translation**) $[1] : \mathscr{T} \to \mathscr{T}$ and a collection of diagrams (called **distinguished triangles**) $X \to Y \to Z \to X[1]$ satisfying the following axioms:
   (a) (Existence) Any morphism can be completed to a distinguished triangle. Any diagram isomorphic to a distinguished triangle is distinguished.
   (b) (Identity) $X \xrightarrow{\mathrm{id}} X \to 0 \to X[1]$ is a distinguished triangle.
   (c) (Rotation) $X \xrightarrow{f} Y \to Z \to X[1]$ is distinguished if and only if $Y \to Z \to X[1] \xrightarrow{-f[1]} Y[1]$ is.
   (d) (Completion) Any commutative square can be completed to a commutative diagram of distinguished triangles.
   (e) (Octahedron) Any commutative triangle can be completed to an "octahedral diagram" involving four distinguished triangles and four commutative triangles.

2. Let $\mathscr{T}$ and $\mathscr{T}'$ be triangulated categories. An additive functor $F : \mathscr{T} \to \mathscr{T}'$ is called a **triangulated functor** if it commutes with [1] and takes distinguished triangles to distinguished triangles.

   Let $\mathcal{A}$ be an abelian category. An additive functor $F : \mathscr{T} \to \mathcal{A}$ is said to be **cohomological** if it takes distinguished triangles to long exact sequences. In other words, given a distinguished triangle $X \to Y \to Z \to X[1]$, the following sequence in $\mathcal{A}$ should be exact:
$$\cdots \to F(Z[-1]) \to F(X) \to F(Y) \to F(Z) \to F(X[1]) \to F(Y[1]) \to F(Z[1]) \to F(X[2]) \to \cdots$$

3. Let $\mathscr{T}$ be a triangulated category. For any $X \in \mathscr{T}$, the functors $\mathrm{Hom}(X, -) : \mathscr{T} \to \mathbb{Z}$-mod and $\mathrm{Hom}(-, X) : \mathscr{T}^{\mathrm{op}} \to \mathbb{Z}$-mod are cohomological.

4. Let $\mathcal{A}$ be an abelian category. Then $\mathrm{Ch}(\mathcal{A})$ is again an abelian category, but the **homotopy category** $\mathrm{K}(\mathcal{A})$ and the **derived category** $\mathrm{D}(\mathcal{A})$, along with their bounded versions $\mathrm{K}^-(\mathcal{A})$, $\mathrm{D}^+(\mathcal{A})$, etc., are triangulated categories. In any of these, a distinguished triangle is defined to be any diagram isomorphic to one of the form $X \xrightarrow{f} Y \to \mathrm{cone}(f) \to X[1]$.

5. The functors $H^0 : \mathrm{K}(\mathcal{A}) \to \mathcal{A}$ and $H^0 : \mathrm{D}(\mathcal{A}) \to \mathcal{A}$ are cohomological.

6. If $F : \mathcal{A} \to \mathcal{B}$ is a left-exact (resp. right-exact) functor of abelian categories and $\mathcal{A}$ has enough injectives (resp. projectives), there is a triangulated functor called the (total) **right derived functor** $RF : \mathrm{D}^+(\mathcal{A}) \to \mathrm{D}^+(\mathcal{B})$ (resp. **left derived functor** $LF : \mathrm{D}^-(\mathcal{A}) \to \mathrm{D}^-(\mathcal{B})$).

7. If $0 \to A \to B \to C \to 0$ is a short exact sequence in $\mathrm{Ch}(\mathcal{A})$ (or in $\mathcal{A}$), then in $\mathrm{D}(\mathcal{A})$, there is a morphism $C \xrightarrow{\delta} A[1]$ such that $A \to B \to C \xrightarrow{\delta} A[1]$ is a distinguished triangle.

8. An object $X \in \mathrm{D}(\mathcal{A})$ is zero if and only if $H^i(X) = 0$ for all i. (The same does **not** hold for morphisms!)

9. For $A, B \in \mathcal{A}$, there is a natural isomorphism $\mathrm{Hom}_{\mathrm{D}(\mathcal{A})}(A, B[n]) \cong \mathrm{Ext}^n(A, B)$. When $n = 0$, this fact tells us that the natural functor $\mathcal{A} \to \mathrm{D}(\mathcal{A})$ is fully faithful, i.e., that $\mathcal{A}$ can be thought of as a full subcategory of $\mathrm{D}(\mathcal{A})$. An object $X \in \mathrm{D}(\mathcal{A})$ belongs to $\mathcal{A}$ if and only if $H^i(X) = 0$ for all $i \neq 0$.

10. **Truncation**: for any $X \in \mathrm{D}(\mathcal{A})$ and any $n \in \mathbb{Z}$, there is a distinguished triangle $\tau_{\leq n} X \to X \to \tau_{\geq n+1} X \to X[1]$ with the following properties:
    (a) $H^i(\tau_{\leq n} X) \to H^i(X)$ is an isomorphism for $i \leq n$, and $H^i(\tau_{\leq n} X) = 0$ for $i \geq n+1$.
    (b) $H^i(X) \to H^i(\tau_{\geq n+1} X)$ is an isomorphism for $i \geq n+1$, and $H^i(\tau_{\geq n+1} X) = 0$ for $i \leq n$.