Math 7260 Spring 2014

Homological Algebra P. Achar

# Problem Set 2

*Due: February 3, 2014*

1. A *filtered abelian group* is an abelian group $M$ together with a fixed collection of subgroups labelled by integers:
$$\cdots \subset F_{-1}M \subset F_0 M \subset F_1 M \subset \cdots \subset M.$$
A *filtered homomorphism* $f : F_\bullet M \to F_\bullet N$ is a homomorphism of abelian groups $f : M \to N$ such that $f(F_n M) \subset F_n N$ for all $n$.
   (a) (Do not hand in) Show that the category **FAb** of filtered abelian groups is additive.
   (b) Show that **FAb** has kernels and cokernels. Also, describe in concrete terms the monic and epic morphisms in this category.
   (c) Show that **FAb** is *not* abelian. (*Hint:* Find a morphism that is both epic and monic, but not an isomorphism.)

2. Let $h : Y \to X$ and $h' : Y' \to X$ be morphisms in a category. A **fiber product** of $h$ and $h'$ (also called a fiber product of $Y$ and $Y'$ over $X$) is an object denoted $Y \times_X Y'$ together with morphisms $p_1 : Y \times_X Y' \to Y$ and $p_2 : Y \times_X Y' \to Y'$ such that the diagram
$$\begin{tikzcd} Y \times_X Y' \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y' \arrow[d, "h'"] \\ Y \arrow[r, "h"'] & X \end{tikzcd}$$
commutes, and such that the appropriate universal property holds. (Formulate the universal property for yourself.) If $X$ is a terminal object, then a fiber product is the same as a product.
   (a) Suppose that the fiber product of $h : Y \to X$ and $h' : Y' \to X$ exists. Prove that if $h$ is epic, then $p_2$ is epic. Similarly, if $h$ is monic, then $p_2$ is monic.
   (b) Prove that in an abelian category, fiber products always exist. (*Hint:* First work out what a fiber product is in some concrete setting, like abelian groups or vector spaces. Then try to describe that construction using only words that make sense in an arbitrary abelian category, such as $\oplus$, kernel, cokernel, etc.)
   (c) Given a fiber product diagram like that above in an abelian category, prove that "$p_1$ and $h'$ have isomorphic kernels." To make this more precise: let $i : K \to Y \times_X Y'$ be a kernel of $p_1$. Then show that $p_2 \circ i : K \to Y'$ is a kernel of $h'$.

3. (Gelfand–Manin, Ex. II.5.1–3) I have been stressing in class that the word "element" does not make sense in a arbitrary abelian category. Nevertheless, "diagram-chasing" proofs in homological algebra *do* work in arbitrary abelian categories. This exercise justifies that fact, by introducing a notion that obeys "diagram-chasing" rules of the sort you are used to.

   Let $\mathcal{A}$ be an abelian category, and let $X$ be an object of $\mathcal{A}$. Define a **schmelement** of $X$ to be an equivalence of pairs $(Y, h)$, where $Y$ is an object of $\mathcal{A}$ and $h : Y \to X$ is a morphism, and the equivalence relation is given by
$$(Y,h) \sim (Y',h') \qquad \text{if} \qquad \begin{array}{l}\text{there is an object } Z \text{ and epic morphisms} \\ u : Z \to Y,\, u' : Z \to Y' \text{ such that } hu = h'u'\end{array}$$
   If $y$ is a schmelement of $X$, we write $y \in_{\text{schm}} X$. Choose a couple of the parts below to hand in.

(a) (Do not hand in) Prove that $\sim$ really is an equivalence relation. It is obvious that it is symmetric and reflexive. To prove that it is transitive, you'll have to use fiber products.

(b) Let $f : X_1 \to X_2$ be a morphism in $\mathcal{A}$, and let $y \in_{\text{schm}} X_1$. Define a schemelement of $X_2$ by the formula $f(y) = (Y, f \circ h)$, where $(Y, h)$ is some representative of $y$. Show that this formula gives a well-defined map
$$\{\text{schmelements of } X_1\} \to \{\text{schmelements of } X_2\}.$$

In the problems below, we write 0 for the schmelement represented by $(0, 0)$. If $y$ is a schmelement represented by $(Y, h)$, we write $-y$ for the schmelement represented by $(Y, -h)$. Throughout, $f : X_1 \to X_2$ is a morphism in $\mathcal{A}$.

(c) $f$ is monic if and only if for any $y \in_{\text{schm}} X_1$, $f(y) = 0$ implies $y = 0$.

(d) $f$ is monic if and only if for $y, y' \in_{\text{schm}} X_1$, $f(y) = f(y')$ implies $y = y'$.

(e) $f$ is epic if and only if for any $y' \in_{\text{schm}} X_2$, there exists a $y \in_{\text{schm}} X_1$ such that $f(y) = y'$.

(f) $f$ is the zero morphism if and only if $f(y) = 0$ for all $y \in_{\text{schm}} X_1$.

(g) A sequence $Y_1 \xrightarrow{f} Y_2 \xrightarrow{g} Y_3$ is exact at $Y_2$ if and only if $g \circ f = 0$ and for any $y' \in_{\text{schm}} Y_2$ with $g(y') = 0$, there exists a $y \in_{\text{schm}} Y_1$ with $f(y) = y'$.

(h) Suppose $y, y' \in_{\text{schm}} X_1$, and that $f(y) = f(y')$. Then there exists $z \in_{\text{schm}} X_1$ such that $g(z) = 0$ and, moreover, for any other morphism $g : Y_1 \to Z$, we have that $g(y) = 0$ implies that $g(z) = -g(y')$, and $g(y') = 0$ implies that $g(z) = g(y)$. (The schmelement $z$ should be thought of as the "difference" $y - y'$.)

4. (Do not hand in—but you **must** do this problem!) Prove the Five Lemma.

5. (Do not hand in—but you **must** do this problem!) Prove the Snake Lemma. Then formulate and prove a version of the snake lemma that incorporates the naturality of the "connecting homomorphism" $\partial$ that we discussed in class. When you are done, you may reward yourself by watching

http://www.youtube.com/watch?v=etbcKWEKnvg.