Problem Set 4

Due: April 11, 2014

1. $\mathcal{A}$ be an abelian category. Show that $X \in \mathcal{A}$ is projective if and only if $\mathrm{Ext}^1(X,Y) = 0$ for all $Y \in \mathcal{A}$. Similarly, show that X is injective if and only if $\mathrm{Ext}^1(Y,X) = 0$ for all Y.

2. (Not to hand in) Let $R = \mathbb{C}[\mathbf{t}, \mathbf{t}^{-1}]$, the ring of Laurent polynomials in one variable. An R-module is the same as a complex vector space V equipped with an automorphism $\mathbf{t} : V \to V$. Given an R-module M, let $M^{\mathbf{t}}$ be the space of $\mathbf{t}$-invariants in M:
$$M^{\mathbf{t}} = \{m \in M \mid \mathbf{t}m = m\}.$$
Given two R-modules M, N, consider the space $\mathrm{Hom}_{\mathbb{C}}(M,N)$ of linear transformations between them. This can be made into an R-module as follows: for $f \in \mathrm{Hom}_{\mathbb{C}}(M,N)$, let $(\mathbf{t} \cdot f)(m) = \mathbf{t}f(\mathbf{t}^{-1}m)$. Show that there is a natural isomorphism
$$\mathrm{Hom}_R(M,N) \cong \mathrm{Hom}_{\mathbb{C}}(M,N)^{\mathbf{t}}.$$

3. Now prove the derived version of the previous result.
   (a) Explain how to define $R\mathrm{Hom}_{\mathbb{C}} : \mathrm{D}^-(R\text{-mod})^{\mathrm{op}} \times \mathrm{D}^+(R\text{-mod}) \to \mathrm{D}^+(R\text{-mod})$.
   (b) Let $J : R\text{-mod} \to \mathbb{C}\text{-mod}$ be the functor $J(M) = M^{\mathbf{t}}$. Show that J is left exact.
   (c) Prove that for $M \in \mathrm{D}^-(R\text{-mod})$ and $N \in \mathrm{D}^+(R\text{-mod})$, there is a natural isomorphism
   $$R\mathrm{Hom}_R(M,N) \cong RJ(R\mathrm{Hom}_{\mathbb{C}}(M,N)).$$
   Hint: You will need to show that if $A, I \in R$-mod with I injective, then $\mathrm{Hom}_{\mathbb{C}}(A,I)$ is also an injective R-module.

   There is nothing particularly special about the ring R here—the same statement is true for any commutative $\mathbb{C}$-algebra. The questions below, however, rely on particular features of R.

4. Show that for any $M \in R$-mod, we have $R^iJ(M) = 0$ for $i \geq 2$. (*Hint:* It might be hard to think about injective resolutions in general. Try converting the question into one that involves a projective resolution instead.) Can you give a concrete description of the functor $R^1J : R\text{-mod} \to \mathbb{C}\text{-mod}$?

5. Show that for $M \in \mathrm{D}^-(R\text{-mod})$, $N \in \mathrm{D}^+(R\text{-mod})$, there is a natural short exact sequence
$$0 \to R^1J(H^{-1}(R\mathrm{Hom}_C(M,N))) \to \mathrm{Hom}_{\mathrm{D}(R\text{-mod})}(M,N) \to J(H^0(R\mathrm{Hom}_C(M,N))) \to 0.$$
Hint: Try to imitate the proof of the universal coefficient theorem that I did in class.