1. Let \( \mathcal{A} \) be an abelian category. Show that \( X \in \mathcal{A} \) is projective if and only if \( \text{Ext}^1(X, Y) = 0 \) for all \( Y \in \mathcal{A} \). Similarly, show that \( X \) is injective if and only if \( \text{Ext}^1(Y, X) = 0 \) for all \( Y \).

2. (Not to hand in) Let \( R = \mathbb{C}[t, t^{-1}] \), the ring of Laurent polynomials in one variable. An \( R \)-module is the same as a complex vector space \( V \) equipped with an automorphism \( t : V \to V \). Given an \( R \)-module \( M \), let \( M^t \) be the space of \( t \)-invariants in \( M \):

\[
M^t = \{ m \in M \mid tm = m \}.
\]

Given two \( R \)-modules \( M, N \), consider the space \( \text{Hom}_C(M, N) \) of linear transformations between them. This can be made into an \( R \)-module as follows: for \( f \in \text{Hom}_C(M, N) \), let \( (t \cdot f)(m) = tf(t^{-1}m) \). Show that there is a natural isomorphism

\[
\text{Hom}_R(M, N) \cong \text{Hom}_C(M, N)^t.
\]

3. Now prove the derived version of the previous result.

   (a) Explain how to define \( \text{RHom}_C : \text{D}^-(R\text{-mod})^\text{op} \times \text{D}^+(R\text{-mod}) \to \text{D}^+(R\text{-mod}) \).

   (b) Let \( J : \text{R-mod} \to \text{C-mod} \) be the functor \( J(M) = M^t \). Show that \( J \) is left exact.

   (c) Prove that for \( M \in \text{D}^-\text{(R-mod)} \) and \( N \in \text{D}^+\text{(R-mod)} \), there is a natural isomorphism

\[
\text{RHom}_R(M, N) \cong RJ(\text{RHom}_C(M, N)).
\]

   \textbf{Hint:} You will need to show that if \( A, I \in \text{R-mod} \) with \( I \) injective, then \( \text{Hom}_C(A, I) \) is also an injective \( R \)-module.

There is nothing particularly special about the ring \( R \) here—the same statement is true for any commutative \( C \)-algebra. The questions below, however, rely on particular features of \( R \).

4. Show that for any \( M \in \text{R-mod} \), we have \( \text{R}^iJ(M) = 0 \) for \( i \geq 2 \). \textbf{(Hint:} It might be hard to think about injective resolutions in general. Try converting the question into one that involves a projective resolution instead.) Can you give a concrete description of the functor \( \text{R}^iJ : \text{R-mod} \to \text{C-mod} \)?

5. Show that for \( M \in \text{D}^-\text{(R-mod)} \), \( N \in \text{D}^+\text{(R-mod)} \), there is a natural short exact sequence

\[
0 \to \text{R}^1J(H^{-1}(\text{RHom}_C(M, N))) \to \text{Hom}_{\text{D}^-(\text{R-mod})}(M, N) \to J(H^0(\text{RHom}_C(M, N))) \to 0.
\]

   \textbf{Hint:} Try to imitate the proof of the universal coefficient theorem that I did in class.