Problem Set 5 (Optional)

Due: May 6, 2014

1. Use a spectral sequence argument to do Problem 5 on Problem Set 4.

2. Let X be a fixed abelian group, and let $F : \mathbb{Z}\text{-mod} \to \mathbb{Z}\text{-mod}$ be the functor $F(M) = \operatorname{Tor}_1(X, M)$. Show that F is left exact, and that there is a natural isomorphism

$$RF[1] \cong X \overset{L}{\otimes} -.$$

Note that this unusual-looking statement is an isomorphism between a left-derived functor and a right-derived functor.

3. (Koszul duality) Let $A = \mathbb{C}[x]/(x^2)$ and $B = \mathbb{C}[y]$. We regard each of these as graded rings by putting $\deg x = -1$ and $\deg y = 1$. Let A-gmod and B-gmod be the abelian categories of finitely-generated graded modules over A and B, respectively. For a graded module $M = \bigoplus M_n$, let $M\langle k\rangle$ be the graded module obtained by shifting the grading by k:

$$(M\langle k\rangle)_n = M_{k+n}.$$

(a) Consider $\mathbb{C}$ as a graded A-module concentrated in degree 0. Explain how to make

$$\bigoplus_{n\geq 0} \operatorname{Ext}^n(\mathbb{C}, \mathbb{C}\langle n\rangle)$$

into a ring. Show that this ring is isomorphic to B. Essentially the same statement (but with $\langle -n\rangle$ instead of $\langle n\rangle$) holds with the roles of A and B reversed.

(b) Let $\mathcal{C} \subset D^b(A\text{-gmod})$ be the full subcategory consisting of objects X with the property that for all $i \in \mathbb{Z}$, the graded A-module $H^i(X)$ is concentrated in degree i. Prove that $\mathcal{C}$ is equivalent to B-gmod. *Hint:* Use the formula

$$X \mapsto \bigoplus_{n\geq 0} \operatorname{Ext}^n(\mathbb{C}, X)$$

to define a functor $\mathcal{C} \to B$-gmod. Again, a similar statement holds with the roles of A and B reversed.

With some more work, the functor $B\text{-gmod} \cong \mathcal{C} \hookrightarrow D^b(A\text{-gmod})$ can be upgraded to an equivalence of categories $D^b(B\text{-gmod}) \overset{\sim}{\to} D^b(A\text{-gmod})$. This situation—where each of A and B arises as an Ext-algebra over the other one, and the two derived categories are equivalent—is called *Koszul duality*. More generally, if A is an exterior algebra on n generators and B is a polynomial ring on n generators, then A and B are Koszul dual to each other.