1. Use a spectral sequence argument to do Problem 5 on Problem Set 4.

2. Let $X$ be a fixed abelian group, and let $F : \mathbb{Z}\text{-mod} \to \mathbb{Z}\text{-mod}$ be the functor $F(M) = \text{Tor}_1(X, M)$. Show that $F$ is left exact, and that there is a natural isomorphism

$$RF[1] \cong X \overset{L}{\otimes}.$$

Note that this unusual-looking statement is an isomorphism between a left-derived functor and a right-derived functor.

3. (Koszul duality) Let $A = \mathbb{C}[x]/(x^2)$ and $B = \mathbb{C}[y]$. We regard each of these as graded rings by putting $\deg x = -1$ and $\deg y = 1$. Let $A\text{-gmod}$ and $B\text{-gmod}$ be the abelian categories of finitely-generated graded modules over $A$ and $B$, respectively. For a graded module $M = \bigoplus M_n$, let $M(k)$ be the graded module obtained by shifting the grading by $k$:

$$(M(k))_n = M_{k+n}.$$

(a) Consider $\mathbb{C}$ as a graded $A$-module concentrated in degree 0. Explain how to make

$$\bigoplus_{n \geq 0} \text{Ext}^n(\mathbb{C}, \mathbb{C}\langle n \rangle)$$

into a ring. Show that this ring is isomorphic to $B$. Essentially the same statement (but with $\langle -n \rangle$ instead of $\langle n \rangle$) holds with the roles of $A$ and $B$ reversed.

(b) Let $\mathcal{C} \subset D^b(A\text{-gmod})$ be the full subcategory consisting of objects $X$ with the property that for all $i \in \mathbb{Z}$, the graded $A$-module $H^i(X)$ is concentrated in degree $i$. Prove that $\mathcal{C}$ is equivalent to $B\text{-gmod}$. Hint: Use the formula

$$X \mapsto \bigoplus_{n \geq 0} \text{Ext}^n(\mathbb{C}, X)$$

to define a functor $\mathcal{C} \to B\text{-gmod}$. Again, a similar statement holds with the roles of $A$ and $B$ reversed.

With some more work, the functor $B\text{-gmod} \cong \mathcal{C} \hookrightarrow D^b(A\text{-gmod})$ can be upgraded to an equivalence of categories $D^b(B\text{-gmod}) \sim D^b(A\text{-gmod})$. This situation—where each of $A$ and $B$ arises as an Ext-algebra over the other one, and the two derived categories are equivalent—is called Koszul duality. More generally, if $A$ is an exterior algebra on $n$ generators and $B$ is a polynomial ring on $n$ generators, then $A$ and $B$ are Koszul dual to each other.