Math 7260 Fall 2015

Homological Algebra P. Achar

Problem Set 3

Due: November 3, 2015

1. Let $\mathcal{A}$ be an abelian category with enough projectives, and let $A \in \mathcal{A}$. Regard A as a chain complex concentrated in degree 0. Choose a projective resolution $P^\bullet \to A$. (Recall that with our new conventions, $P^\bullet$ looks like $\cdots \to P^{-2} \to P^{-1} \to P^0 \to 0$.) Show that for any object $B^\bullet$ in $D(\mathcal{A})$ and any morphism $f : A \to B^\bullet$, f can be represented by a roof of the form

$$\begin{tikzcd} & P^\bullet \arrow[dl] \arrow[dr] & \\ A & & B^\bullet \end{tikzcd}$$

2. Now, assume $B \in \mathcal{A}$. Using the previous problem, show that

$$\mathrm{Hom}_{D(\mathcal{A})}(A, B[n]) \cong \begin{cases} 0 & \text{if } n < 0, \\ \mathrm{Ext}^n(A, B) & \text{if } n \geq 0. \end{cases}$$

   Remark. Here is a generalization of the "$n < 0$" case above. (This isn't a homework problem; it relies on facts about $D(\mathcal{A})$ that we haven't gotten to yet.) Let $D(\mathcal{A})^{\leq n}$ denote the subcategory of $D(\mathcal{A})$ consisting of objects $X^\bullet$ such that $H^i(X^\bullet) = 0$ for $i > n$. Similarly, let $D(\mathcal{A})^{\geq n}$ be the subcategory consisting of $X^\bullet$ such that $H^i(X^\bullet) = 0$ for $i < n$. If $X^\bullet \in D(\mathcal{A})^{\leq n}$ and $Y^\bullet \in D(\mathcal{A})^{\geq n+1}$, then

$$\mathrm{Hom}_{D(\mathcal{A})}(X^\bullet, Y^\bullet) = 0.$$

   Convince yourself that the "$n < 0$" case of the problem is indeed a special case of this fact.

3. Use the fact that $\mathrm{Ext}^1(A, B) \cong \mathrm{Hom}_{D(\mathcal{A})}(A, B[1])$ to give a new proof of the fact that

$$\mathrm{Ext}^1(A, B) \cong \{\text{equivalence classes of short exact sequences } 0 \to B \to X \to A \to 0\}.$$