Lecture 1 (Jan. 14) Quiver: $Q = (Q_0, Q_1)$. $\text{Rep}_k(Q)$: category of finite-dimensional representations of $Q$ over a field $k$. For $s \in Q_0$, let $V_s$ be the representation with $k$ at vertex $s$ and 0's elsewhere. The $V_s$'s are the simple objects of $\text{Rep}_k(Q)$.

Each $M \in \text{Rep}_k(Q)$ has a dimension vector $\dim M : Q_0 \to \mathbb{Z}_{\geq 0}$. Given $d : Q_0 \to \mathbb{Z}_{\geq 0}$, form the moduli space of representations with dimension vector $d$:

$$\text{Rep}_d(Q) := \prod_{(s \to t) \in Q_1} \text{Hom}(k^{d(s)}, k^{d(t)}).$$

This is acted on by the groups

$$G_d := \prod_{s \in Q_0} \text{GL}_{d(s)}(k)$$

and

$$\bar{G}_d := G_d/(\text{diagonal copy of } k^\times).$$

The $G_d$-action on $\text{Rep}_d(Q)$ factors through $\bar{G}_d$. There is a natural bijection

$$\{G_d\text{- or } \bar{G}_d\text{-orbits on } \text{Rep}_d(Q)\} \leftrightarrow \{\text{isom. classes of reps of } Q \text{ with dim vector } d\}.$$

Lecture 2 (Jan. 16) Rouquier notes §3.3: ADE classification of graphs with positive-definite quadratic form. The quadratic form of an undirected graph $\Gamma = (\Gamma_0, \Gamma_1)$ is defined to be the form $q$ on $R^{\Gamma_0}$ given by

$$q\left(\sum_{s \in \Gamma_0} x_s s\right) = \sum_{s \in \Gamma_0} x_s^2 - \sum_{(s \to t) \in \Gamma_1} x_s x_t.$$

For an ADE quiver, define a positive root to be a vector $x = \sum x_s s$ with $x_s \in \mathbb{Z}_{\geq 0}$ such that $q(x) = 1$.

Lectures 3–4 (Jan. 21, 23) Rouquier notes §3.4: Gabriel’s Theorem. $\text{Rep}_k(Q)$ contains finitely many isomorphism classes of indecomposables if and only if $Q$ is an ADE quiver. In that case, the isomorphism classes of indecomposables are in bijection with the positive roots.

Proof sketch ($\Rightarrow$). If $Q$ has finitely many indecomposables, then the quadratic form $q$ of the underlying graph is positive-definite, so $Q$ is an ADE quiver.

Proof sketch ($\Leftarrow$). If $V$ is indecomposable and has dimension vector $d$, then $V$ corresponds to a dense orbit in $\text{Rep}_d(Q)$, and $q(d) = 1$. Since there are only finitely many positive roots, there are only finitely many indecomposables.

Both parts of the proof use the following Euler form (recall that $\text{Ext}^{\geq 2} = 0$ on $\text{Rep}(Q)$):

$$\langle V, W \rangle = \sum_{r \geq 0} (-1)^r \text{Ext}^r(V, W) = \dim \text{Hom}(V, W) - \dim \text{Ext}^1(V, W),$$

along with the explicit formula

$$\langle V, W \rangle = \sum_{s \in Q_0} (\dim V_s)(\dim W_s) - \sum_{(s \to t) \in Q_1} (\dim V_s)(\dim W_t).$$
For an ADE quiver, we denote the indecomposable objects of $\text{Rep}_k(Q)$ as follows:

$$\{\text{positive roots}\} \leftrightarrow \{\text{indecomposables}\}$$

$$\gamma \leftrightarrow M(\gamma)$$

If $\gamma$ is a vector corresponding to a vertex $t$ (i.e., $\gamma = \sum x_s s$ with $x_s = \delta_{st}$), then $M(\gamma)$ is the simple representation $V_t$.

**Lecture 5  (Jan. 26)** Rouquier notes §3.2: Hall algebra of a finitary category. If $\mathcal{A}$ is finitary (i.e. all Hom- and Ext$^1$-groups are finite sets), then $\mathcal{H}_\mathcal{A}$ is the $\mathbb{C}$-vector space with basis $\{\text{isomorphism classes of objects of } \mathcal{A}\}$, and multiplication given by

$$[M] \ast [N] = \sum_{[L]} F^L_{M,N}[L]$$

where $F^L_{M,N}$ is the number of subobjects $L' \subset L$ such that $L' \cong N$ and $L/L' \cong M$.

**Example.** Let $\mathcal{A}$ be the category of finite-dimensional vector spaces over a finite field $k = \mathbb{F}_q$. Then $[k^n] \ast [k^m] = \binom{m+n}{m}_q [k^{m+n}]$ and $[k]^n = [n]_q ![k^n]$.

Here the $q$-binomial coefficient and the $q$-factorial use the quantum integers $[n]_q := q^n - 1 - q^{-n}$.

**Lectures 6–7  (Jan. 28, 30)** General background on abstract root systems, semisimple Lie algebras, and quantum groups. Notational convention from now on:

$\Phi$ is root system, $\Phi^+ \subset \Phi$ is the set of positive roots, $\Delta \subset \Phi^+$ is the set of simple roots

Given the data $\Phi \supset \Phi^+ \supset \Delta$, the associated generic quantum group is

$$U_v(\mathfrak{g}) : \text{a } \mathbb{Q}(v)\text{-algebra generated by } K^\pm_\alpha, E_\alpha, F_\alpha, \text{ with } \alpha \in \Delta.$$ 

Its definition involves the $v$-quantum numbers $[n]_v = \frac{v^n - v^{-n}}{v - v^{-1}}$. The Lusztig integral form is

$$U_{L,v}(\mathfrak{g}) : \text{the } \mathbb{Z}[v, v^{-1}]\text{-subalgebra of } U_v(\mathfrak{g}) \text{ generated by } K^\pm_\alpha \text{ and by the "divided powers"}$$

$$E^{(n)}_\alpha := \frac{E^n_\alpha}{[n]_v^!}, \quad F^{(n)}_\alpha := \frac{F^n_\alpha}{[n]_v^!}.$$ 

Sometimes, it is convenient to define $U_{L,v}$ as a $\mathbb{Q}[v, v^{-1}]$-algebra instead. (There is also a De Concini–Kac integral form that we will not study.) Given $q \in \mathbb{C}^\times$, the specialized quantum group is defined to be

$$U_q(\mathfrak{g}) = \mathbb{C} \otimes_{\mathbb{Z}[v, v^{-1}]} U_{L,v}(\mathfrak{g}),$$

where $\mathbb{C}$ is made into a $\mathbb{Z}[v, v^{-1}]$-module via $v \mapsto q$. The positive parts

$$U^+_v(\mathfrak{g}), \quad U^+_{L,v}(\mathfrak{g}), \quad U^+_q(\mathfrak{g})$$

are defined to be the subalgebras generated by just the $E_\alpha$’s (or by the $E^{(n)}_\alpha$’s in the latter two cases).
Lectures 8–12 (Feb. 2–11) Ringel’s Theorem. Let $\Phi \supset \Phi^+ \supset \Delta$ be a simply-laced root system. Let $Q$ be a quiver whose underlying graph is the Dynkin diagram of that root system. In particular, the set of vertices $Q_0$ is identified with $\Delta$. Let $k = \mathbb{F}_q$. Endow the Hall algebra $H_{\text{Rep}_k(Q)}$ with a modified product:

\[ [M] \cdot [N] = q^{(M,N)/2} [M] * [N]. \]

Ringel’s Theorem. There is a unique ring isomorphism $U_{\sqrt{q}}(g) \sim \rightarrow (H_{\text{Rep}_k(Q)}, \cdot)$ sending $E_\alpha$ to $[V_\alpha]$.

In the proof, we need notation for arbitrary objects of $\text{Rep}(Q)$. By Gabriel’s theorem, each $\gamma \in \Phi^+$ corresponds to an indecomposable object $M(\gamma)$. (If $\gamma \in \Delta$, then $M(\gamma) \cong V_\gamma$ is simple.) So an arbitrary representation has the form $M(n) := \bigoplus_{\gamma \in \Phi^+} M(\gamma)^{n(\gamma)}$ for some $n : \Phi^+ \to \mathbb{Z}_{\geq 0}$.

Proof sketch. See Rouquier’s notes §4.2.3.

1. Construct a ring homomorphism $\phi : U_{\sqrt{q}}(g) \rightarrow (H_{\text{Rep}_k(Q)}, \cdot)$. This comes down to checking that the $[V_\alpha]$’s satisfy the quantum Serre relations. That, in turn, is a calculation in the Hall algebra of the quiver $\bullet \rightarrow \bullet$.

2. Show that $\Phi^+$ admits an ordering $\{\gamma_1, \ldots, \gamma_N\}$ so that $\text{Hom}(M(\gamma_i), M(\gamma_j)) = 0$ if $j < i$. On the other hand, $\text{Ext}^1(M(\gamma_i), M(\gamma_j)) = 0$ if $j \geq i$.

3. Let $\alpha_1, \ldots, \alpha_n$ be the order on $\Delta$ induced by that on $\Phi^+$. Then

\[ [M(n)] = [M(\gamma_1)^{n(\gamma_1)}] * \cdots * [M(\gamma_N)^{n(\gamma_N)}] \quad \text{and} \quad [M(\alpha_1)^{d(\alpha_1)}] * \cdots * [M(\alpha_n)^{d(\alpha_n)}] = \sum_{n : \Phi^+ \to \mathbb{Z}_{\geq 0}} \sum_{\dim(n) = d} [M(n)]. \]

4. Use the previous step to show that $\phi$ is surjective. Then, equip both $U_{\sqrt{q}}(g)$ and $H_{\text{Rep}_k(Q)}$ with a grading by $|\Delta| = \{d : \Delta \to \mathbb{Z}\}$ by declaring $\deg E_\alpha = \dim V_\alpha$, $\deg[M] = \dim M$.

The map $\phi$ preserves this grading. Moreover, corresponding graded components of the two algebras have the same dimension:

\[ \dim U_{\sqrt{q}}(g)^d = \left| \left\{ n : \Phi^+ \rightarrow \mathbb{Z}_{\geq 0} \mid \dim(n) = d \right\} \right| = \dim(H_{\text{Rep}_k(Q)})^d. \]

So $\phi$ is an isomorphism.

Lectures 13–14 (Feb. 13, 18) Lusztig’s geometric Hall algebra. Some new notation:

\[
\{ \text{G}_d\text{-orbits on } \text{Rep}_d(Q) \} \leftrightarrow \{ \text{reps of } Q \text{ of dimension } d \} \leftrightarrow \{ n : \Phi^+ \rightarrow \mathbb{Z}_{\geq 0} \mid \dim(n) = d \}
\]

Let $F_d(Q) = \text{space of } G_d\text{-invariant functions } \text{Rep}_d(Q) \rightarrow \mathbb{C}$.
and let $\chi_n : \text{Rep}_{d'}(Q) \to \mathbb{C}$ be the characteristic function of $O_n$. Clearly \{\chi_n \ | \ \dim(n) = d\} is a basis for $F_d(Q)$. There is a vector space isomorphism

$$F(Q) := \bigoplus_{d' : \Delta \to \mathbb{Z}_{\geq 0}} F_{d'}(Q) \cong \mathcal{H}(\text{Rep}(Q))$$

that sends $\chi_n \mapsto [M(n)]$. \hfill (1)

We will now give $F(Q)$ a ring structure that makes this into a ring isomorphism. Let $d', d'' : \Delta \to \mathbb{Z}_{\geq 0}$ be two dimension vectors, and let $d = d' + d''$.

Any point $\phi = (\phi_{\alpha \to \beta})_{(\alpha, \beta) \in Q_1} \in \text{Rep}_{d'}(Q)$ determines a representation $M(\phi)$ of $Q$ with $\dim M(\phi) = d'$. Recall that a point $g = (g_{\alpha})_{\alpha \in \Delta}$ acts on $\phi$ by $g \cdot \phi = (g_{\beta \phi_{\alpha \to \beta} g_{\alpha}^{-1}})$. Consider the diagram

$$\text{Rep}_{d'}(Q) \times \text{Rep}_{d''}(Q) \xrightarrow{\times} \text{E}'' \xrightarrow{\varphi} \text{E}'' \xrightarrow{\psi''} \text{Rep}_{d''}(Q)$$

where the intermediate spaces are

$$E'' = \{(\phi, M) \ | \ \phi \in \text{Rep}_{d'}(Q), M \subset M(\phi) \text{ a subrepresentation of dimension } d'\}
$$

$$\mathcal{E}'' = \{(\phi_{\alpha \to \beta}), (M_{\alpha}) | (\phi_{\alpha \to \beta}) \in \text{Rep}_{d'}(Q), M_{\alpha} \subset M_{\beta}, \dim M_{\alpha} = d''(\alpha), \phi_{\alpha \to \beta}(M_{\alpha}) \subset M_{\beta}\},$$

and the maps are given by

$$\psi' = (\psi'_{\alpha \to \beta}) \in \text{Rep}_{d'}(Q) \text{ and } \psi'' = (\psi''_{\alpha \to \beta}) \in \text{Rep}_{d''}(Q) \text{ are given by}$$

$$\psi'_{\alpha \to \beta} : k^{d''(\alpha)}(\mu_{\alpha \to \beta}'^{-1}) \xrightarrow{\phi_{\alpha \to \beta}} k^{d''(\beta)}(\mu_{\alpha \to \beta}'^{-1}) \text{ and } \psi''_{\alpha \to \beta} : k^{d''(\alpha)}(\mu_{\alpha \to \beta}'^{-1}) \xrightarrow{\phi_{\alpha \to \beta}} k^{d''(\beta)}(\mu_{\alpha \to \beta}')$$

Group actions: we have $G_{d'} \times G_{d'} \curvearrowright \text{Rep}_{d'}(Q) \times \text{Rep}_{d''}(Q)$ and $G_d \curvearrowright \text{Rep}_{d}(Q)$ as usual, and

$$G_{d'} \times G_{d''} \times G_d \curvearrowright E', \quad (g', g'', g) \cdot (\phi, M, \mu', \mu') = ((g_{\beta \phi_{\alpha \to \beta} g_{\alpha}^{-1}}), (g_{\alpha}M_{\alpha}), (g_{\alpha \mu_{\alpha \to \beta} g_{\alpha}^{-1}})),$$

$$G_d \curvearrowright E''$$

Given a vector space $V$, let $\mathcal{B}(V)$ be the space of all bases for it. $GL(V)$ acts on $\mathcal{B}(V)$ freely and transitively. Let $Gr(r, n)$ denote the Grassmannian of $r$-dimensional subspaces of $k^n$. The fibers of $\varpi$ and $\varpi'$ are smooth:

$$\varpi^{-1}(\psi', \psi'') \cong \prod_{\alpha \in \Delta} \text{Gr}(d''(\alpha), d(\alpha)) \times \prod_{\alpha \in \Delta} \mathcal{B}(M_{\alpha}) \times \prod_{\alpha \in \Delta} \mathcal{B}(k^{d''(\alpha)}/M_{\alpha}) \times \prod_{(\alpha \to \beta) \in Q_1} k^{d''(\alpha)d'(\beta)},$$

$$(\varpi')^{-1}(\phi, M) \cong \prod_{\alpha \in \Delta} \mathcal{B}(M_{\alpha}) \times \prod_{\alpha \in \Delta} \mathcal{B}(k^{d''(\alpha)}/M_{\alpha})$$

$G_d$ preserves the fibers of $\varpi$. $G_{d'} \times G_{d''}$ acts freely on $E'$, and $\varpi'$ is the quotient map by that action. Convolutions of functions. Given $f' \in F_{d'}(Q), f'' \in F_{d''}(Q)$, let $f' \ast f'' : \text{Rep}_{d'}(Q) \times \text{Rep}_{d''}(Q) \to \mathbb{C}$ be the function $(\psi' \ast \psi'') \mapsto f'(\psi') f''(\psi'')$. Below, we use “pullback” notation for functions: “$\mu \circ h$” simply means $h \circ \mu$. We define the convolution product $f' \ast f'' \in F_d(Q)$ by

$$(f' \ast f'')(\phi) = \sum_{(\phi, M) \in (\varpi')^{-1}(\phi)} \tilde{f}(\phi, M) \text{ where } \tilde{f} : E'' \to \mathbb{C} \text{ is the unique function}$$

such that $\varpi' \circ \tilde{f} = \varpi'(f' \ast f'')$.

Just by unwinding the definitions, one can show that

$$\chi_{n'} \ast \chi_{n''} = \sum_{n : n \oplus \leftrightarrow \mathbb{Z}_{\geq 0}} F_{M(n'), M(n'')} \chi_n,$$

making (1) into a ring isomorphism.
**Addendum.** There is another way to modify the Hall algebra from the one discussed earlier. Let $M$ and $N$ be representations of the quiver $Q$, and let $d' = \dim M$ and $d'' = \dim N$. Define a new pairing as follows:

$$\langle M, N \rangle = \sum_{\alpha \in \Delta} (\dim M_\alpha)(\dim N_\alpha) + \sum_{(\alpha \rightarrow \beta) \in Q_1} (\dim M_\alpha)(\dim N_\beta) = \sum_{\alpha \in \Delta} d'(\alpha)d''(\alpha) + \sum_{(\alpha \rightarrow \beta) \in Q_1} d'(\alpha)d''(\beta).$$

(This differs from the Euler form by the sign on the second term.) Let $\circ$ be the product given by

$$[M] \circ [N] = q^{-\langle M, N \rangle/2}[M] \ast [N].$$

**Ringel’s Theorem Redux.** There is a unique ring isomorphism

$$U^+_{\sqrt{q}}(g) \sim (\mathcal{H}_{\text{Rep}_k(Q)}, \circ) \quad \text{sending } E_\alpha \text{ to } [V_\alpha].$$

The proof is the same as before; one just needs to do a Hall algebra computation using $\circ$ instead of $\cdot$.

The two versions of Ringel’s theorem together imply that there is a ring isomorphism

$$(\mathcal{H}_{\text{Rep}_k(Q)}, \cdot) \sim (\mathcal{H}_{\text{Rep}_k(Q)}, \circ).$$

This cannot be the identity map (although it does send $[V_\alpha]$ to itself).

**Proposition.** The ring isomorphism $(\mathcal{H}_{\text{Rep}_k(Q)}, \cdot) \sim (\mathcal{H}_{\text{Rep}_k(Q)}, \circ)$ is given by

$$[M(n)] \mapsto q^{(-\dim C_n - \dim \text{End}(M(n)) + \dim M(n))/2}[M(n)],$$

where $\dim M(n)$ means the total dimension $\sum_{\alpha \in \Delta} \dim M(n)_\alpha$.

**Proof.** As a first step, we claim that if $d = \dim(n)$, then

$$-\dim C_n - \dim \text{End}(M(n)) + \dim M(n) = \sum_{\alpha \in \Delta} d(\alpha) - \sum_{\alpha \in \Delta} d(\alpha)^2. \quad (3)$$

Let $\gamma_1, \ldots, \gamma_N$ be the ordering on $\Phi^+$ that came up in the proof of Ringel’s theorem. Write $M(n)$ as $\bigoplus_{i=1}^N M(\gamma_i)_{n(\gamma_i)}$. Let $d_i = \dim M(\gamma_i)_{n(\gamma_i)}$, so $d = \sum_{i=1}^N d_i$. We have

$$\text{End}(M(n)) \cong \bigoplus_{i,j} \text{Hom}(M(\gamma_i)_{n(\gamma_i)}, M(\gamma_j)_{n(\gamma_j)}) \cong \bigoplus_{i \leq j} \text{Hom}(M(\gamma_i)_{n(\gamma_i)}, M(\gamma_j)_{n(\gamma_j)}).$$

Since $\text{Ext}^1(M(\gamma_i), M(\gamma_j)) = 0$ for $i \leq j$, we have

$$\dim \text{End} M(n) = \sum_{i \leq j} (\dim \text{Hom}(M(\gamma_i)_{n(\gamma_i)}, M(\gamma_j)_{n(\gamma_j)})) - \dim \text{Ext}^1(M(\gamma_i)_{n(\gamma_i)}, M(\gamma_j)_{n(\gamma_j)})$$

$$= \sum_{i \leq j} (d_i, d_j) = \sum_{i \leq j} d_i(\alpha)d_j(\alpha) - \sum_{(\alpha \rightarrow \beta) \in Q_1} d_i(\alpha)d_j(\beta).$$

On the other hand, according to [1, Proposition 6.6], we have

$$\dim C_n = \sum_{i \leq j} d_i(\alpha)d_j(\alpha) + \sum_{(\alpha \rightarrow \beta) \in Q_1} d_i(\alpha)d_j(\beta).$$

Note that the first sum has $i < j$, while the second has $i \leq j$. Note also that $\sum_{i \leq j} d_i(\alpha)d_j(\alpha) = \sum_{i < j; \alpha} d_j(\alpha)d_i(\alpha) = \sum_{i > j; \alpha} d_i(\alpha)d_j(\alpha)$. Therefore,
Let \( \text{Intersection cohomology/perverse sheaves.} \)

For any \( F \in \mathcal{P} \) extensions. That is, for any distinguished triangle \( F \xrightarrow{\delta} G \xrightarrow{\epsilon} H \xrightarrow{\mu} F \)

This proves (3).

Now let \( n', n'' : \Phi^+ \rightarrow \mathbb{Z}_{\geq 0} \). Let \( d' = \text{dim}(n') \) and \( d'' = \text{dim}(n'') \), and let \( d = d' + d'' \). Using (3), we see that the proposition is equivalent to the assertion that

\[
q^{\sum d' + (d'' - d')^2/2} [M(n')] \circ q^{\sum d'' + (d' - d'')^2/2} [M(n'')]
\]

By the definition of \( \circ \), the left-hand side above is equal to

\[
q^{\sum d'} [M(n')] \cdot q^{\sum d''} [M(n'')]
\]

Comparing the coefficients of \( [M(n)] \), the proposition reduces to showing that

\[
\sum d' - \sum (d')^2 + \sum d'' - \sum (d'')^2 - \langle d', d'' \rangle = |d'| + |d''| - \sum d - \sum (d - 1)^2.
\]

An easy calculation with the definitions of \( \langle - , - \rangle \) and \( \langle - , - \rangle \) shows that this is the case.

Lectures 15–16  (Feb. 20, 23)  Background on sheaves, derived categories of sheaves, sheaf functors.  
Intersection cohomology/perverse sheaves. Let \( X = \bigsqcup_{s \in J} X_s \) be a stratified algebraic variety. For each stratum \( X_s \), there is a unique object \( \text{IC}(X_s) \) characterized by the following properties:

- \( \text{IC}(X_s) \) is supported on \( \overline{X}_s \).
- \( \text{IC}(X_s)|_{X_s} \cong C_{X_s}[\text{dim } X_s] \).
- For any \( X_t \subset \overline{X}_s, X_t \neq X_s, \mathcal{H}^i(\text{IC}(X_s)|_{X_t}) = 0 \) for \( i \geq - \text{dim } X_t \).
- The preceding property also holds for \( \mathcal{D}(\text{IC}(X_s)) \).

There is a slight generalization of this of this notion that sometimes comes up: for any local system (i.e., locally constant sheaf) \( \mathcal{L} \) on \( X_s \), there is a unique object \( \text{IC}(X_s, \mathcal{L}) \), characterized as above, except that \( \text{IC}(X_s, \mathcal{L})|_{X_s} \cong \mathcal{L}[\text{dim } X_s] \). (But nontrivial local systems do not occur on Lusztig quiver varieties.)

Let \( P(X) \subset D^b(X) \) be the smallest full subcategory that contains all the \( \text{IC}(X_s, \mathcal{L}) \)'s and is closed under extensions. That is, for any distinguished triangle \( \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow \mathcal{F}' \). Then \( \mathcal{F} \in P(X) \). Objects of \( P(X) \) are called perverse sheaves. Key properties of this category:

- It is an abelian category.
- Every object has a composition series. The \( \text{IC}(X_s, \mathcal{L}) \)'s (for \( \mathcal{L} \) an irreducible local system) are precisely the simple objects in this category.
- If \( \mu \) is a smooth morphism of varieties of relative dimension \( r \), then \( \mu^*[r] \cong \mu^![-r] \) takes perverse sheaves to perverse sheaves.
- Call an object of \( D^b(X) \) semisimple if it is a direct sum of various \( \text{IC}(X_s, \mathcal{L})[n_i] \)'s. The Decomposition Theorem asserts that if \( \mu \) is a proper morphism of varieties, then \( R\mu_\ast \) takes semisimple objects to semisimple objects.
Lectures 17–18  (Feb. 25, 27) Convolution of sheaves. Given $F' \in D^b_{\text{Dr}}(\text{Rep}_d(Q))$, $F'' \in D^b_{\text{Dr}}(\text{Rep}_d'(Q))$, define the convolution product $F' \ast F'' \in D^b_{\text{Dr}}(\text{Rep}_d(Q))$ by

$$F' \ast F'' = R\alpha_\ast \tilde{F}$$

where $\tilde{F} \in D^b_{\text{Dr}}(E'')$ is the unique object such that $\varpi^\ast \tilde{F} = \varpi^\ast (F' \boxtimes F'')$.

Let $j_n : \mathcal{O}_n \hookrightarrow \text{Rep}_d(Q)$ be the inclusion map, and let $\Xi_n := j_n^\ast \mathcal{O}_n$. Also, for a point $\phi \in \text{Rep}_d(Q)$, let

$$F^M_{M(\phi'),M(\phi'')} = \text{the variety of submodules } M \subset M(\phi) \text{ such that } M \cong M(n') \text{ and } M(\phi)/M \cong M(n').$$

The sheaf-theoretic analogue of (2) is this: the stalks of $\Xi_{n'} \ast \Xi_{n''}$ are given by

$$(\Xi_{n'} \ast \Xi_{n''})|_\phi \cong R\Gamma(F^M_{M(n'),M(n'')}).$$

For a smooth morphism of varieties $\mu$ of relative dimension $r$, recall that $\mu^! \cong \mu^* [2r]$, and that the functor $\mu^! = \mu^* [r] \cong \mu^! [r]^{-r}$ takes perverse sheaves to perverse sheaves. Define the modified convolution product $F' \circ F'' \in D^b_{\text{Dr}}(\text{Rep}_d(Q))$ by

$$F' \circ F'' = R\varpi_\ast \tilde{F}$$

where $\tilde{F} \in D^b_{\text{Dr}}(E'')$ is the unique object such that $\varpi^\ast \tilde{F} = \varpi^\ast (F' \boxtimes F'')$.

From the earlier dimension calculations, one finds that

$$F' \circ F'' \cong (F' \ast F'')[[\langle d',d'' \rangle]].$$

There is a version $K_d$ of the Grothendieck group of $D^b_{\text{Dr}}(\text{Rep}_d(Q))$ that is a module over $\mathbb{Z}[v,v^{-1}]$, with the property that $[\text{IC}(\mathcal{O}_n)[n]] = v^{-n} [\text{IC}(\mathcal{O}_n)]$. The equation above implies that if $\mathcal{O}_{n'} \subset \text{Rep}_d'(Q)$ and $\mathcal{O}_{n''} \subset \text{Rep}_d''(Q)$, then in $K_d$,

$$[\text{IC}(\mathcal{O}_{n'}) \circ \text{IC}(\mathcal{O}_{n''})] = v^{-\langle d',d'' \rangle} [\text{IC}(\mathcal{O}_{n'}) \ast \text{IC}(\mathcal{O}_{n''})].$$

Using nontrivial facts about mixed sheaves on $\text{Rep}_d(Q)$, one can show that in $K_d$,

$$[\text{IC}(\mathcal{O}_n)] = v^{-\dim \mathcal{O}_n} [\Xi_n] + \sum_{n' : \mathcal{O}_{n''} \subset \mathcal{O}_n ; \ n' \neq n} v^{-\dim \mathcal{O}_{n''} + 1} \text{rk } H^{\dim \mathcal{O}_{n''} + 1} (IC(\mathcal{O}_n)|_{\mathcal{O}_{n''}})[\Xi_{n''}]$$

(4)

The restriction $i < 0$ is written here for emphasis, but it is not strictly necessary: the terms in the sum vanish for $i \geq 0$ anyway, by one of the defining properties of IC objects.

Theorem (Lusztig). There is an isomorphism of $\mathbb{Z}[v,v^{-1}]$-modules

$$K_d \cong \bigoplus_{d : \Delta \to \mathbb{Z} \geq 0} K_d \cong U^+_L(\mathfrak{g})$$

(5)

such that we have the following behavior of products and objects:

$$K \xrightarrow{\sim} U^+_L(\mathfrak{g}) \xrightarrow{v^{-\sqrt{q}}} U^+_{\sqrt{q}}(\mathfrak{g}) \xrightarrow{\text{Ringel} \circ} \text{Rep}_d(Q) \xrightarrow{\sim} F(Q)$$

$$\circ \quad \text{mult.} \quad \text{mult.} \quad \circ$$

$$[\Xi_n] \xrightarrow{\circ \text{-lift of } [M(n)]] \xrightarrow{\ast} [M(n)] \xrightarrow{\chi_n}$$

Here, the “$\circ$-lifts of $[M(n)]$” are the elements whose existence you proved in Problem Set 2, Exercise 5. (They are called “$\circ$-lifts” because we are using the $\circ$ version of Ringel’s isomorphism; the $\cdot$ version would give different elements.)
The formula (4) highlights the importance of the elements \( v^{-\dim \mathcal{O}_n} \Xi_n \). Let us introduce the elements

\[
\mathcal{E}_n := v^{-\dim \mathcal{O}_n}(\varnothing\text{-lift of } [M(n)]) = \text{image of } v^{-\dim \mathcal{O}_n} \Xi_n \text{ under Eq. (5)}.
\]

The elements \( \{\mathcal{E}_n\} \) form the PBW-type basis of \( U_{L,v}^+(\mathfrak{g}) \).

**Canonical basis.** Let \( \mathcal{B}_n \) be the image of \( [\mathcal{I}(\mathcal{O}_n)] \) under (5). These elements form a new basis of \( U_{L,v}^+(\mathfrak{g}) \), called the canonical basis. This basis has two special properties: (i) **Positivity:** the Decomposition Theorem implies that

\[
\mathcal{B}_n \mathcal{B}_n' = \sum m_{n,n'} \mathcal{B}_{n'} \quad \text{where each } m_{n,n'} \in \mathbb{Z}[v,v^{-1}] \text{ is } -\text{-stable and has nonnegative coefficients.}
\]

(ii) **Independence of } Q:** If \( Q' \) is another quiver with the same underlying undirected graph as \( Q \), there is an equivalence of derived categories \( D_b^{\bullet} \text{(Rep}_d(Q)) \cong D_b^{\bullet} \text{(Rep}_d(Q')) \) given by a kind of Fourier transform. This equivalence commutes with \( \varnothing \)-convolution, and it sends IC’s to IC’s (although it does not preserve the labels of orbits). See [1, §13].

The basis elements \( \mathcal{B}_n \) can be computed explicitly, using an algebraic analogues of the characterization of the IC(\( \mathcal{O}_n \))’s. First, an analogue of Verdier duality: define an involution

\[
\varnothing : U_{L,v}^+(\mathfrak{g}) \to U_{L,v}^+(\mathfrak{g}) \quad \text{by} \quad \mathcal{E}_n = \mathcal{E}_n (\alpha \in \Delta), \quad v = v^{-1}.
\]

Let us now translate (4) across the isomorphism (5). For each \( n : \Phi^+ \to \mathbb{Z}_{\geq 0}, \mathcal{B}_n \) is the unique element of \( U_{L,v}^+(\mathfrak{g}) \) such that

\[
\mathcal{B}_n = \mathcal{E}_n + \sum_{m : \Phi^+ \to \mathbb{Z}_{\geq 0}, \mathcal{O}_{m \subset \mathcal{O}_n}, \mathcal{O}_m} P_{n,m} \mathcal{E}_m \quad \text{with } P_{n,m} \in v^{-1} \mathbb{Z}[v^{-1}], \quad \text{and} \quad \overline{\mathcal{B}_n} = \mathcal{B}_n.
\]

If we write them in the \( \{[M(n)]\} \) basis (or, to be precise, in the \( \{\varnothing\text{-lift of } [M(n)]\} \) basis) instead, we see that \( \mathcal{B}_n \) is uniquely characterized by the properties that

\[
\mathcal{B}_n = v^{-\dim \mathcal{O}_n}[M(n)] + \sum_{m : \Phi^+ \to \mathbb{Z}_{\geq 0}, \mathcal{O}_{m \subset \mathcal{O}_n}, \mathcal{O}_m} P_{n,m}[M(m)] \quad \text{with } P_{n,m} \in v^{-\dim \mathcal{O}_m - 1} \mathbb{Z}[v^{-1}], \quad \text{and} \quad \overline{\mathcal{B}_n} = \mathcal{B}_n.
\]

**Aside.** If we use the \( \cdot \) version of Ringel’s theorem to lift the \( [M(n)] \)'s instead, the setup looks like this:

\[
\begin{array}{c}
K \xrightarrow{\sim} U_{L,v}^+(\mathfrak{g}) \xrightarrow{v^{-1/2}} U_{L,v}^+(\mathfrak{g}) \xrightarrow{\sim} \mathcal{H}_{\text{Rep}(Q)}
\
\overset{\varnothing}{\longrightarrow} \xrightarrow{\cdot} v^{\dim \mathcal{O}_n \cdot \text{End}(M(n))^{-\dim M(n)}(-\text{lift of } [M(n)])}
\xrightarrow{\cdot} \mathcal{E}_n = v^{\dim \mathcal{O}_n \cdot \text{End}(M(n))^{-\dim M(n)}(-\text{lift of } [M(n)])}
\end{array}
\]

This version matches Theorems 4.11 and 4.12 from Rouquier’s notes. Note that both the PBW basis \( \{\mathcal{E}_n\} \) and the canonical basis \( \{\mathcal{B}_n\} \) are the same as above, but the \( \{\text{lift of } [M(n)]\} \) basis is different.

Many sources seem to use the \( \cdot \) version, which looks really unnatural to me from the geometric perspective. Lusztig doesn’t use either \( \cdot \) or \( \varnothing \); instead, he uses \( \ast \), with a slightly different version of Ringel’s theorem that incorporates the elements \( K_{\alpha}^\pm \in U_{L,v}(\mathfrak{g}) \).

8
Pop Quiz

1. What are the definitions of $Q^w$, $\hat{\varphi}$, and $\hat{\theta}$?

2. Some boxes have assumptions like $\theta \cdot \varphi = 0$ or $\lambda \cdot \varphi = 0$, and others don’t. Explain these assumptions.

3. Some boxes use the action of $G_\varphi$, and others use $G_\varphi$. Explain why.
References