Lecture Notes

Lecture 1 *(Jan. 14)* Quiver: $Q = (Q_0, Q_1)$. $\mathrm{Rep}_k(Q)$: category of finite-dimensional representations of Q over a field k. For $s \in Q_0$, let V_s be the representation with k at vertex s and 0's elsewhere. The V_s's are the simple objects of $\mathrm{Rep}_k(Q)$.

Each $M \in \mathrm{Rep}_k(Q)$ has a *dimension vector* $\underline{\dim}\, M : Q_0 \to \mathbb{Z}_{\geq 0}$. Given $\mathbf{d} : Q_0 \to \mathbb{Z}_{\geq 0}$, form the moduli space of representations with dimension vector $\mathbf{d}$:

$$\mathrm{Rep}_{\mathbf{d}}(Q) := \prod_{(s \to t) \in Q_1} \mathrm{Hom}(k^{\mathbf{d}(s)}, k^{\mathbf{d}(t)}).$$

This is acted on by the groups

$$G_{\mathbf{d}} := \prod_{s \in Q_0} GL_{\mathbf{d}(s)}(k) \qquad \text{and} \qquad \bar{G}_{\mathbf{d}} := G_{\mathbf{d}}/(\text{diagonal copy of } k^\times).$$

The $G_{\mathbf{d}}$-action on $\mathrm{Rep}_{\mathbf{d}}(Q)$ factors through $\bar{G}_{\mathbf{d}}$. There is a natural bijection

$$\{G_{\mathbf{d}}\text{- or } \bar{G}_{\mathbf{d}}\text{-orbits on } \mathrm{Rep}_{\mathbf{d}}(Q)\} \longleftrightarrow \{\text{isom. classes of reps of } Q \text{ with dim vector } \mathbf{d}\}$$

Lecture 2 *(Jan. 16)* Rouquier notes §3.3: ADE classification of graphs with positive-definite quadratic form. The quadratic form of an undirected graph $\Gamma = (\Gamma_0, \Gamma_1)$ is defined to be the form q on $\mathbb{R}\Gamma_0$ given by

$$q\left(\sum_{s \in \Gamma_0} x_s s\right) = \sum_{s \in \Gamma_0} x_s^2 - \sum_{(s - t) \in \Gamma_1} x_s x_t.$$

For an ADE quiver, define a *positive root* to be a vector $\mathbf{x} = \sum x_s s$ with $x_s \in \mathbb{Z}_{\geq 0}$ such that $q(\mathbf{x}) = 1$.

Lectures 3–4 *(Jan. 21, 23)* Rouquier notes §3.4: Gabriel's Theorem. $\mathrm{Rep}_k(Q)$ contains finitely many isomorphism classes of indecomposables if and only if Q is an ADE quiver. In that case, the isomorphism classes of indecomposables are in bijection with the *positive roots*.

Proof sketch ($\Longrightarrow$). If Q has finitely many indecomposables, then the quadratic form q of the underlying graph is positive-definite, so Q is an ADE quiver.

Proof sketch ($\Longleftarrow$). If V is indecomposable and has dimension vector $\mathbf{d}$, then V corresponds to a dense orbit in $\mathrm{Rep}_{\mathbf{d}}(Q)$, and $q(\mathbf{d}) = 1$. Since there are only finitely many positive roots, there are only finitely many indecomposables.

Both parts of the proof use the following *Euler form* (recall that $\mathrm{Ext}^{\geq 2} = 0$ on $\mathrm{Rep}(Q)$):

$$\langle V, W \rangle = \sum_{r \geq 0} (-1)^r \mathrm{Ext}^r(V, W) = \dim \mathrm{Hom}(V, W) - \dim \mathrm{Ext}^1(V, W),$$

along with the explicit formula

$$\langle V, W \rangle = \sum_{s \in Q_0} (\dim V_s)(\dim W_s) - \sum_{(s \to t) \in Q_1} (\dim V_s)(\dim W_t).$$

For an ADE quiver, we denote the indecomposable objects of $\mathrm{Rep}_k(Q)$ as follows:

$$\{\text{positive roots}\} \longleftrightarrow \{\text{indecomposables}\}$$
$$\gamma \longleftrightarrow M(\gamma)$$

If γ is a vector corresponding to a vertex t (i.e., $\gamma = \sum x_s s$ with $x_s = \delta_{st}$), then $M(\gamma)$ is the simple representation V_t.

Lecture 5 *(Jan. 26)* Rouquier notes §3.2: Hall algebra of a finitary category. If $\mathcal{A}$ is finitary (i.e. all Hom- and Ext^1-groups are finite sets), then $\mathcal{H}_{\mathcal{A}}$ is the $\mathbb{C}$-vector space with basis {isomorphism classes of objects of $\mathcal{A}$}, and multiplication given by

$$[M] * [N] = \sum_{[L]} F^L_{M,N}[L]$$

where $F^L_{M,N}$ is the number of subobjects $L' \subset L$ such that $L' \cong N$ and $L/L' \cong M$.
Example. Let $\mathcal{A}$ be the category of finite-dimensional vector spaces over a finite field $k = \mathbb{F}_q$. Then

$$[k^n] * [k^m] = \binom{m+n}{m}_q [k^{m+n}] \qquad \text{and} \qquad [k]^n = [n]_q![k^n].$$

Here the q-binomial coefficient and the q-factorial use the *quantum integers* $[n]_q := \frac{q^n-1}{q-1}$.

Lectures 6–7 *(Jan. 28, 30)* General background on abstract root systems, semisimple Lie algebras, and quantum groups. Notational convention from now on:

Φ is root system, $\qquad \Phi^+ \subset \Phi$ is the set of positive roots, $\qquad \Delta \subset \Phi^+$ is the set of simple roots

Given the data $\Phi \supset \Phi^+ \supset \Delta$, the associated *generic quantum group* is

$$U_v(\mathfrak{g}) : \text{a } \mathbb{Q}(v)\text{-algebra generated by } K_\alpha^{\pm 1},\ E_\alpha,\ F_\alpha, \text{ with } \alpha \in \Delta.$$

Its definition involves the v-quantum numbers $[\![n]\!]_v = \frac{v^n - v^{-n}}{v - v^{-1}}$. The *Lusztig integral form* is

$$U_{\mathrm{L},v}(\mathfrak{g}) : \text{the } \mathbb{Z}[v, v^{-1}]\text{-subalgebra of } U_v(\mathfrak{g}) \text{ generated by } K_\alpha^{\pm 1} \text{ and by the “divided powers”}$$
$$E_\alpha^{(n)} := \frac{E_\alpha^n}{[\![n]\!]_v!}, \qquad F_\alpha^{(n)} := \frac{F_\alpha^n}{[\![n]\!]_v!}.$$

Sometimes, it is convenient to define $U_{\mathrm{L},v}$ as a $\mathbb{Q}[v, v^{-1}]$-algebra instead. (There is also a *De Concini–Kac integral form* that we will not study.) Given $q \in \mathbb{C}^\times$, the *specialized quantum group* is defined to be

$$U_q(\mathfrak{g}) = \mathbb{C} \otimes_{\mathbb{Z}[v,v^{-1}]} U_{\mathrm{L},v}(\mathfrak{g}),$$

where $\mathbb{C}$ is made into a $\mathbb{Z}[v, v^{-1}]$-module via $v \mapsto q$. The *positive parts*

$$U_v^+(\mathfrak{g}), \qquad U_{\mathrm{L},v}^+(\mathfrak{g}), \qquad U_q^+(\mathfrak{g})$$

are defined to be the subalgebras generated by just the E_α's (or by the $E_\alpha^{(n)}$'s in the latter two cases).

Lectures 8–12 *(Feb. 2–11)* Ringel's Theorem. Let $\Phi \supset \Phi^+ \supset \Delta$ be a simply-laced root system. Let Q be a quiver whose underlying graph is the Dynkin diagram of that root system. In particular, the set of vertices Q_0 is identified with Δ. Let $k = \mathbb{F}_q$. Endow the Hall algebra $\mathcal{H}_{\mathrm{Rep}_k(Q)}$ with a modified product:

$$[M] \cdot [N] = q^{\langle M,N \rangle/2}[M] * [N].$$

Ringel's Theorem. There is a unique ring isomorphism

$$U^+_{\sqrt{q}}(\mathfrak{g}) \xrightarrow{\sim} (\mathcal{H}_{\mathrm{Rep}_k(Q)}, \cdot) \qquad \text{sending } E_\alpha \text{ to } [V_\alpha].$$

In the proof, we need notation for arbitrary objects of $\mathrm{Rep}(Q)$. By Gabriel's theorem, each $\gamma \in \Phi^+$ corresponds to an indecomposable object $M(\gamma)$. (If $\gamma \in \Delta$, then $M(\gamma) \cong V_\gamma$ is simple.) So an arbitrary representation has the form

$$M(\mathbf{n}) := \bigoplus_{\gamma \in \Phi^+} M(\gamma)^{\oplus \mathbf{n}(\gamma)} \qquad \text{for some} \qquad \mathbf{n} : \Phi^+ \to \mathbb{Z}_{\geq 0}.$$

Proof sketch. See Rouquier's notes §4.2.3.

1. Construct a ring homomorphism $\phi : U^+_{\sqrt{q}}(\mathfrak{g}) \to (\mathcal{H}_{\mathrm{Rep}_k(Q)}, \cdot)$. This comes down to checking that the $[V_\alpha]$'s satisfy the quantum Serre relations. That, in turn, is a calculation in the Hall algebra of the quiver $\bullet \longrightarrow \bullet$.

2. Show that Φ^+ admits an ordering $\{\gamma_1, \ldots, \gamma_N\}$ so that $\mathrm{Hom}(M(\gamma_i), M(\gamma_j)) = 0$ if $j < i$. On the other hand, $\mathrm{Ext}^1(M(\gamma_i), M(\gamma_j)) = 0$ if $j \geq i$.

3. Let $\alpha_1, \ldots, \alpha_n$ be the order on Δ induced by that on Φ^+. Then

$$[M(\mathbf{n})] = [M(\gamma_1)^{\mathbf{n}(\gamma_1)}] * \cdots * [M(\gamma_N)^{\mathbf{n}(\gamma_N)}] \quad \text{and} \quad [M(\alpha_n)^{\mathbf{d}(\alpha_n)}] * \cdots * [M(\alpha_1)^{\mathbf{d}(\alpha_1)}] = \sum_{\substack{\mathbf{n} : \Phi^+ \to \mathbb{Z}_{\geq 0} \\ \underline{\dim}(\mathbf{n}) = \mathbf{d}}} [M(\mathbf{n})].$$

4. Use the previous step to show that ϕ is surjective. Then, equip both $U^+_v(\mathfrak{g})$ and $\mathcal{H}_{\mathrm{Rep}_k(Q)}$ with a grading by $\mathbb{Z}^{|\Delta|} = \{\mathbf{d} : \Delta \to \mathbb{Z}\}$ by declaring

$$\deg E_\alpha = \underline{\dim}\, V_\alpha, \qquad \deg[M] = \underline{\dim}\, M.$$

The map ϕ preserves this grading. Moreover, corresponding graded components of the two algebras have the same dimension:

$$\dim U^+_v(\mathfrak{g})^{\mathbf{d}} = |\{\mathbf{n} : \Phi^+ \to \mathbb{Z}_{\geq 0} \mid \underline{\dim}(\mathbf{n}) = \mathbf{d}\}| = \dim(\mathcal{H}_{\mathrm{Rep}_k(Q)})^{\mathbf{d}}.$$

So ϕ is an isomorphism.

Lectures 13–14 *(Feb. 13, 18)* Lusztig's geometric Hall algebra. Some new notation:

$$\begin{array}{ccccc} \{G_{\mathbf{d}}\text{-orbits on } \mathrm{Rep}_{\mathbf{d}}(Q)\} & \longleftrightarrow & \{\text{reps of } Q \text{ of dimension } \mathbf{d}\} & \longleftrightarrow & \{\mathbf{n} : \Phi^+ \to \mathbb{Z}_{\geq 0} \mid \underline{\dim}(\mathbf{n}) = \mathbf{d}\} \\ \mathcal{O}_{\mathbf{n}} & \longleftarrow & \longrightarrow & & \mathbf{n} \end{array}$$

Let

$$F_{\mathbf{d}}(Q) = \text{space of } G_{\mathbf{d}}\text{-invariant functions } \mathrm{Rep}_{\mathbf{d}}(Q) \to \mathbb{C},$$

and let $\chi_{\mathbf{n}} : \mathrm{Rep}_{\mathbf{d}}(Q) \to \mathbb{C}$ be the characteristic function of $\mathcal{O}_{\mathbf{n}}$. Clearly $\{\chi_{\mathbf{n}} \mid \underline{\dim}(\mathbf{n}) = \mathbf{d}\}$ is a basis for $F_{\mathbf{d}}(Q)$. There is a vector space isomorphism

$$F(Q) := \bigoplus_{\mathbf{d}:\Delta\to\mathbb{Z}_{\geq 0}} F_{\mathbf{d}}(Q) \xrightarrow{\sim} \mathcal{H}_{\mathrm{Rep}(Q)} \qquad \text{that sends } \chi_{\mathbf{n}} \mapsto [M(\mathbf{n})]. \tag{1}$$

We will now give $F(Q)$ a ring structure that makes this into a ring isomorphism. Let $\mathbf{d}', \mathbf{d}'' : \Delta \to \mathbb{Z}_{\geq 0}$ be two dimension vectors, and let $\mathbf{d} = \mathbf{d}' + \mathbf{d}''$.

Any point $\phi = (\phi_{\alpha\to\beta})_{(\alpha\to\beta)\in Q_1} \in \mathrm{Rep}_{\mathbf{d}}(Q)$ determines a representation $M(\phi)$ of Q with $\underline{\dim}\, M(\phi) = \mathbf{d}$. Recall that a point $g = (g_\alpha)_{\alpha\in\Delta}$ acts on ϕ by $g \cdot \phi = (g_\beta \phi_{\alpha\to\beta} g_\alpha^{-1})$. Consider the diagram

$$\mathrm{Rep}_{\mathbf{d}'}(Q) \times \mathrm{Rep}_{\mathbf{d}''}(Q) \xleftarrow{\varpi} \mathbf{E}' \xrightarrow{\varpi'} \mathbf{E}'' \xrightarrow{\varpi''} \mathrm{Rep}_{\mathbf{d}}(Q)$$

where the intermediate spaces are

$$\begin{aligned}
\mathbf{E}'' &= \{(\phi, M) \mid \phi \in \mathrm{Rep}_{\mathbf{d}}(Q),\ M \subset M(\phi) \text{ a subrepresentation of dimension } \mathbf{d}'\} \\
&= \{((\phi_{\alpha\to\beta}), (M_\alpha)) \mid (\phi_{\alpha\to\beta}) \in \mathrm{Rep}_{\mathbf{d}}(Q),\ M_\alpha \subset k^{\mathbf{d}(\alpha)},\ \dim M_\alpha = \mathbf{d}''(\alpha),\ \phi_{\alpha\to\beta}(M_\alpha) \subset M_\beta\}, \\
\mathbf{E}' &= \{(\phi, M, \mu'', \mu') \mid (\phi, M) \in \mathbf{E}'',\ \mu'' \text{ a choice of basis for } M,\ \mu' \text{ a choice of basis for } M(\phi)/M\} \\
&= \{((\phi_{\alpha\to\beta}), (M_\alpha), (\mu''_\alpha), (\mu'_\alpha)) \mid ((\phi_{\alpha\to\beta}), (M_\alpha)) \in \mathbf{E}'',\ \mu''_\alpha : M_\alpha \xrightarrow{\sim} k^{\mathbf{d}''(\alpha)},\ \mu'_\alpha : k^{\mathbf{d}(\alpha)}/M_\alpha \xrightarrow{\sim} k^{\mathbf{d}'(\alpha)}\},
\end{aligned}$$

and the maps are given by

$$(\psi', \psi'') \overset{\varpi}{\leftarrow\!\shortmid} (\phi, M, \mu'', \mu') \overset{\varpi'}{\mapsto} (\phi, M) \overset{\varpi''}{\mapsto} \phi$$

where $\psi' = (\psi'_{\alpha\to\beta}) \in \mathrm{Rep}_{\mathbf{d}'}(Q)$ and $\psi'' = (\psi''_{\alpha\to\beta}) \in \mathrm{Rep}_{\mathbf{d}''}(Q)$ are given by

$$\psi'_{\alpha\to\beta} : k^{\mathbf{d}'(\alpha)} \xrightarrow{(\mu'_\alpha)^{-1}} \frac{k^{\mathbf{d}(\alpha)}}{M_\alpha} \xrightarrow{\phi_{\alpha\to\beta}} \frac{k^{\mathbf{d}(\beta)}}{M_\beta} \xrightarrow{\mu'_\beta} k^{\mathbf{d}'(\beta)}, \qquad \psi'' : k^{\mathbf{d}''(\alpha)} \xrightarrow{(\mu''_\alpha)^{-1}} M_\alpha \xrightarrow{\phi_{\alpha\to\beta}} M_\beta \xrightarrow{\mu''_\beta} k^{\mathbf{d}''(\beta)}$$

Group actions: we have $G_{\mathbf{d}'} \times G_{\mathbf{d}'} \curvearrowright \mathrm{Rep}_{\mathbf{d}'}(Q) \times \mathrm{Rep}_{\mathbf{d}''}(Q)$ and $G_{\mathbf{d}} \curvearrowright \mathrm{Rep}_{\mathbf{d}}(Q)$ as usual, and

$$\begin{aligned}
G_{\mathbf{d}'} \times G_{\mathbf{d}''} \times G_{\mathbf{d}} \curvearrowright \mathbf{E}' \qquad & (g', g'', g) \cdot (\phi, M, \mu'', \mu') = ((g_\beta \phi_{\alpha\to\beta} g_\alpha^{-1}), (g_\alpha M_\alpha), (g''_\alpha \mu''_\alpha g_\alpha^{-1}), (g'_\alpha \mu'_\alpha g_\alpha^{-1})), \\
G_{\mathbf{d}} \curvearrowright \mathbf{E}'' \qquad & g \cdot (\phi, M) = ((g_\beta \phi_{\alpha\to\beta} g_\alpha^{-1}), (g_\alpha M_\alpha)).
\end{aligned}$$

Given a vector space V, let $\mathbb{B}(V)$ be the space of all bases for it. $GL(V)$ acts on $\mathbb{B}(V)$ freely and transitively. Let $\mathrm{Gr}(r, n)$ denote the Grassmannian of r-dimensional subspaces of k^n. The fibers of ϖ and ϖ' are smooth:

$$\varpi^{-1}(\psi', \psi'') \cong \prod_{\alpha\in\Delta} \mathrm{Gr}(\mathbf{d}''(\alpha), \mathbf{d}(\alpha)) \times \prod_{\alpha\in\Delta} \mathbb{B}(M_\alpha) \times \prod_{\alpha\in\Delta} \mathbb{B}(k^{\mathbf{d}(\alpha)}/M_\alpha) \times \prod_{(\alpha\to\beta)\in Q_1} \mathbb{A}^{\mathbf{d}'(\alpha)\mathbf{d}''(\beta)},$$

$$(\varpi')^{-1}(\phi, M) \cong \prod_{\alpha\in\Delta} \mathbb{B}(M_\alpha) \times \prod_{\alpha\in\Delta} \mathbb{B}(k^{\mathbf{d}(\alpha)}/M_\alpha)$$

$G_{\mathbf{d}}$ preserves the fibers of ϖ. $G_{\mathbf{d}'} \times G_{\mathbf{d}''}$ acts freely on $\mathbf{E}'$, and ϖ' is the quotient map by that action.

Convolution of functions. Given $f' \in F_{\mathbf{d}'}(Q)$, $f'' \in F_{\mathbf{d}''}(Q)$, let $f' \times f'' : \mathrm{Rep}_{\mathbf{d}'}(Q) \times \mathrm{Rep}_{\mathbf{d}''}(Q) \to \mathbb{C}$ be the function $(\psi', \psi'') \mapsto f'(\psi') f''(\psi'')$. Below, we use "pullback" notation for functions: "$\mu^* h$" simply means $h \circ \mu$. We define the convolution product $f' * f'' \in F_{\mathbf{d}}(Q)$ by

$$(f' * f'')(\phi) = \sum_{(\phi, M)\in(\varpi'')^{-1}(\phi)} \tilde{f}(\phi, M) \qquad \begin{array}{l}\text{where } \tilde{f} : \mathbf{E}'' \to \mathbb{C} \text{ is the unique function} \\ \quad \text{such that } \varpi'^* \tilde{f} = \varpi^*(f' \times f'').\end{array}$$

Just by unwinding the definitions, one can show that

$$\chi_{\mathbf{n}'} * \chi_{\mathbf{n}''} = \sum_{\mathbf{n}:\Phi^+\to\mathbb{Z}_{\geq 0}} F^{M(\mathbf{n})}_{M(\mathbf{n}'), M(\mathbf{n}'')} \chi_{\mathbf{n}}, \tag{2}$$

making (1) into a ring isomorphism.

Addendum. There is another way to modify the Hall algebra from the one discussed earlier. Let M and N be representations of the quiver Q, and let $\mathbf{d}' = \underline{\dim}\, M$ and $\mathbf{d}'' = \underline{\dim}\, N$. Define a new pairing as follows:

$$\langle\!\langle M, N \rangle\!\rangle = \sum_{\alpha \in \Delta} (\dim M_\alpha)(\dim N_\alpha) + \sum_{(\alpha \to \beta) \in Q_1} (\dim M_\alpha)(\dim N_\beta) = \sum_{\alpha \in \Delta} \mathbf{d}'(\alpha)\mathbf{d}''(\alpha) + \sum_{(\alpha \to \beta) \in Q_1} \mathbf{d}'(\alpha)\mathbf{d}''(\beta).$$

(This differs from the Euler form by the sign on the second term.) Let $\diamond$ be the product given by

$$[M] \diamond [N] = q^{-\langle\!\langle M,N \rangle\!\rangle/2} [M] * [N].$$

Ringel's Theorem Redux. There is a unique ring isomorphism

$$U^+_{\sqrt{q}}(\mathfrak{g}) \xrightarrow{\sim} (\mathcal{H}_{\mathrm{Rep}_k(Q)}, \diamond) \qquad \text{sending } E_\alpha \text{ to } [V_\alpha].$$

The proof is the same as before; one just needs to do a Hall algebra computation using $\diamond$ instead of $\cdot$.
The two versions of Ringel's theorem together imply that there is a ring isomorphism

$$(\mathcal{H}_{\mathrm{Rep}_k(Q)}, \cdot) \xrightarrow{\sim} (\mathcal{H}_{\mathrm{Rep}_k(Q)}, \diamond).$$

This cannot be the identity map (although it does send $[V_\alpha]$ to itself).
Proposition. The ring isomorphism $(\mathcal{H}_{\mathrm{Rep}_k(Q)}, \cdot) \xrightarrow{\sim} (\mathcal{H}_{\mathrm{Rep}_k(Q)}, \diamond)$ is given by

$$[M(\mathbf{n})] \mapsto q^{(-\dim \mathcal{O}_\mathbf{n} - \dim \mathrm{End}(M(\mathbf{n})) + \dim M(\mathbf{n}))/2} [M(\mathbf{n})],$$

where $\dim M(\mathbf{n})$ means the total dimension $\sum_{\alpha \in \Delta} \dim M(\mathbf{n})_\alpha$.
Proof. As a first step, we claim that if $\mathbf{d} = \underline{\dim}(\mathbf{n})$, then

$$-\dim \mathcal{O}_\mathbf{n} - \dim \mathrm{End}(M(\mathbf{n})) + \dim M(\mathbf{n}) = \sum_{\alpha \in \Delta} \mathbf{d}(\alpha) - \sum_{\alpha \in \Delta} \mathbf{d}(\alpha)^2. \tag{3}$$

Let $\gamma_1, \ldots, \gamma_N$ be the ordering on Φ^+ that came up in the proof of Ringel's theorem. Write $M(\mathbf{n})$ as $\bigoplus_{i=1}^N M(\gamma_i)^{\mathbf{n}(\gamma_i)}$. Let $\mathbf{d}_i = \underline{\dim}\, M(\gamma_i)^{\mathbf{n}(\gamma_i)}$, so $\mathbf{d} = \sum_{i=1}^N \mathbf{d}_i$. We have

$$\mathrm{End}(M(\mathbf{n})) \cong \bigoplus_{i,j} \mathrm{Hom}(M(\gamma_i)^{\mathbf{n}(\gamma_i)}, M(\gamma_j)^{\mathbf{n}(\gamma_j)}) = \bigoplus_{i \le j} \mathrm{Hom}(M(\gamma_i)^{\mathbf{n}(\gamma_i)}, M(\gamma_j)^{\mathbf{n}(\gamma_j)}).$$

Since $\mathrm{Ext}^1(M(\gamma_i), M(\gamma_j)) = 0$ for $i \le j$, we have

$$\begin{aligned} \dim \mathrm{End}\, M(\mathbf{n}) &= \sum_{i \le j} (\dim \mathrm{Hom}(M(\gamma_i)^{\mathbf{n}(\gamma_i)}, M(\gamma_j)^{\mathbf{n}(\gamma_j)}) - \dim \mathrm{Ext}^1(M(\gamma_i)^{\mathbf{n}(\gamma_i)}, M(\gamma_j)^{\mathbf{n}(\gamma_j)}) \\ &= \sum_{i \le j} \langle \mathbf{d}_i, \mathbf{d}_j \rangle = \sum_{\substack{i \le j \\ \alpha \in \Delta}} \mathbf{d}_i(\alpha)\mathbf{d}_j(\alpha) - \sum_{\substack{i \le j \\ (\alpha \to \beta) \in Q_1}} \mathbf{d}_i(\alpha)\mathbf{d}_j(\beta). \end{aligned}$$

On the other hand, according to [1, Proposition 6.6], we have

$$\dim \mathcal{O}_\mathbf{n} = \sum_{\substack{i < j \\ \alpha \in \Delta}} \mathbf{d}_i(\alpha)\mathbf{d}_j(\alpha) + \sum_{\substack{i \le j \\ (\alpha \to \beta) \in Q_1}} \mathbf{d}_i(\alpha)\mathbf{d}_j(\beta).$$

Note that the first sum has $i < j$, while the second has $i \le j$. Note also that $\sum_{i<j;\alpha} \mathbf{d}_i(\alpha)\mathbf{d}_j(\alpha) = \sum_{i<j;\alpha} \mathbf{d}_j(\alpha)\mathbf{d}_i(\alpha) = \sum_{i>j;\alpha} \mathbf{d}_i(\alpha)\mathbf{d}_j(\alpha)$. Therefore,

$$\dim \mathcal{O}_{\mathbf{n}} + \dim \operatorname{End}(M(\mathbf{n})) = \sum_{\substack{i>j \\ \alpha\in\Delta}} \mathbf{d}_i(\alpha)\mathbf{d}_j(\alpha) + \sum_{\substack{i\le j \\ (\alpha\to\beta)\in Q_1}} \mathbf{d}_i(\alpha)\mathbf{d}_j(\beta) + \sum_{\substack{i\le j \\ \alpha\in\Delta}} \mathbf{d}_i(\alpha)\mathbf{d}_j(\alpha) - \sum_{\substack{i\le j \\ (\alpha\to\beta)\in Q_1}} \mathbf{d}_i(\alpha)\mathbf{d}_j(\beta)$$

$$= \sum_{i,j,\alpha} \mathbf{d}_i(\alpha)\mathbf{d}_j(\alpha) = \sum_{\alpha\in\Delta} \left(\sum_{i=1}^{N} \mathbf{d}_i(\alpha)\right)\left(\sum_{j=1}^{N} \mathbf{d}_j(\alpha)\right) = \sum_{\alpha\in\Delta} \mathbf{d}(\alpha)^2.$$

This proves (3).

Now let $\mathbf{n}', \mathbf{n}'' : \Phi^+ \to \mathbb{Z}_{\ge 0}$. Let $\mathbf{d}' = \underline{\dim}(\mathbf{n}')$ and $\mathbf{d}'' = \underline{\dim}(\mathbf{n}'')$, and let $\mathbf{d} = \mathbf{d}' + \mathbf{d}''$. Using (3), we see that the proposition is equivalent to the assertion that

$$q^{(\sum \mathbf{d}'(\alpha) - \sum \mathbf{d}'(\alpha)^2)/2}[M(\mathbf{n}')] \diamond q^{(\sum \mathbf{d}''(\alpha) - \sum \mathbf{d}''(\alpha)^2)/2}[M(\mathbf{n}'')]$$
$$= q^{\langle \mathbf{d}', \mathbf{d}''\rangle/2} \sum F^{M(\mathbf{n})}_{M(\mathbf{n}'),M(\mathbf{n}'')} q^{(\sum \mathbf{d}(\alpha) - \sum \mathbf{d}(\alpha)^2)/2}[M(\mathbf{n})]$$

By the definition of $\diamond$, the left-hand side above is equal to

$$q^{(\sum \mathbf{d}'(\alpha) - \sum \mathbf{d}'(\alpha)^2 + \sum \mathbf{d}''(\alpha) - \sum \mathbf{d}''(\alpha)^2)/2} q^{-\langle\!\langle \mathbf{d}', \mathbf{d}''\rangle\!\rangle/2} \sum F^{M(\mathbf{n})}_{M(\mathbf{n}'),M(\mathbf{n}'')}[M(\mathbf{n})].$$

Comparing the coefficients of $[M(\mathbf{n})]$, the proposition reduces to showing that

$$\sum \mathbf{d}'(\alpha) - \sum \mathbf{d}'(\alpha)^2 + \sum \mathbf{d}''(\alpha) - \sum \mathbf{d}''(\alpha)^2 - \langle\!\langle \mathbf{d}', \mathbf{d}''\rangle\!\rangle = \langle \mathbf{d}', \mathbf{d}''\rangle + \sum \mathbf{d}(\alpha) - \sum \mathbf{d}(\alpha)^2.$$

An easy calculation with the definitions of $\langle -, -\rangle$ and $\langle\!\langle -, -\rangle\!\rangle$ shows that this is the case. $\square$

Lectures 15–16 *(Feb. 20, 23)* Background on sheaves, derived categories of sheaves, sheaf functors.

Intersection cohomology/perverse sheaves. Let $X = \bigsqcup_{s\in J} X_s$ be a stratified algebraic variety. For each stratum X_s, there is a unique object $\mathrm{IC}(X_s)$ characterized by the following properties:

- $\mathrm{IC}(X_s)$ is supported on $\overline{X_s}$.
- $\mathrm{IC}(X_s)|_{X_s} \cong \underline{C}_{X_s}[\dim X_s]$.
- For any $X_t \subset \overline{X_s}$, $X_t \ne X_s$, $\mathcal{H}^i(\mathrm{IC}(X_s)|_{X_t}) = 0$ for $i \ge -\dim X_t$.
- The preceding property also holds for $\mathbb{D}(\mathrm{IC}(X_s))$.

There is a slight generalization of this of this notion that sometimes comes up: for any *local system* (i.e., locally constant sheaf) $\mathcal{L}$ on X_s, there is a unique object $\mathrm{IC}(X_s, \mathcal{L})$, characterized as above, except that $\mathrm{IC}(X_s, \mathcal{L})|_{X_s} \cong \mathcal{L}[\dim X_s]$. (But nontrivial local systems do not occur on Lusztig quiver varieties.)

Let $P(X) \subset D^{\mathrm{b}}(X)$ be the smallest full subcategory that contains all the $\mathrm{IC}(X_s, \mathcal{L})$'s and is closed under extensions. That is, for any distinguished triangle $\mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to$ in $D^{\mathrm{b}}(X)$, if $\mathcal{F}', \mathcal{F}'' \in P(X)$, then $\mathcal{F} \in P(X)$. Objects of $P(X)$ are called *perverse sheaves.* Key properties of this category:

- It is an abelian category.
- Every object has a composition series. The $\mathrm{IC}(X_s, \mathcal{L})$'s (for $\mathcal{L}$ an irreducible local system) are precisely the simple objects in this category.
- If μ is a smooth morphism of varieties of relative dimension r, then $\mu^*[r] \cong \mu^![-r]$ takes perverse sheaves to perverse sheaves.
- Call an object of $D^{\mathrm{b}}(X)$ *semisimple* if it is a direct sum of various $\mathrm{IC}(X_s, \mathcal{L})[n]$'s. The *Decomposition Theorem* asserts that if μ is a proper morphism of varieties, then $R\mu_*$ takes semisimple objects to semisimple objects.

Lectures 17–18 *(Feb. 25, 27)* Convolution of sheaves. Given $\mathcal{F}' \in D^{\mathrm{b}}_{G_{\mathbf{d}'}}(\mathrm{Rep}_{\mathbf{d}'}(Q))$, $\mathcal{F}'' \in D^{\mathrm{b}}_{G''_{\mathbf{d}}}(\mathrm{Rep}_{\mathbf{d}''}(Q))$, define the convolution product $\mathcal{F}' * \mathcal{F}'' \in D^{\mathrm{b}}_{G_{\mathbf{d}}}(\mathrm{Rep}_{\mathbf{d}}(Q))$ by

$$\mathcal{F}' * \mathcal{F}'' = R\varpi''_* \tilde{\mathcal{F}} \qquad \text{where } \tilde{\mathcal{F}} \in D^{\mathrm{b}}_{G_{\mathbf{d}}}(\mathbf{E}'') \text{ is the unique object such that } \varpi'^* \tilde{\mathcal{F}} = \varpi^*(\mathcal{F}' \boxtimes \mathcal{F}'').$$

Let $j_{\mathbf{n}} : \mathcal{O}_{\mathbf{n}} \hookrightarrow \mathrm{Rep}_{\mathbf{d}}(Q)$ be the inclusion map, and let $\Xi_{\mathbf{n}} := j_{\mathbf{n}!}\underline{C}_{\mathcal{O}_{\mathbf{n}}}$. Also, for a point $\phi \in \mathrm{Rep}_{\mathbf{d}}(Q)$, let

$$\mathbf{F}^{M(\phi)}_{M(\mathbf{n}'),M(\mathbf{n}'')} = \text{the variety of submodules } M \subset M(\phi) \text{ such that } M \cong M(\mathbf{n}'') \text{ and } M(\phi)/M \cong M(\mathbf{n}').$$

The sheaf-theoretic analogue of (2) is this: the stalks of $\Xi_{\mathbf{n}'} * \Xi_{\mathbf{n}''}$ are given by

$$(\Xi_{\mathbf{n}'} * \Xi_{\mathbf{n}''})|_\phi \cong R\Gamma(\mathbf{F}^{M(\phi)}_{M(\mathbf{n}'),M(\mathbf{n}'')}).$$

For a smooth morphism of varieties μ of relative dimension r, recall that $\mu^! \cong \mu^*[2r]$, and that the functor $\mu^\sharp = \mu^*[r] \cong \mu^![-r]$ takes perverse sheaves to perverse sheaves. Define the *modified convolution product* $\mathcal{F}' \diamond \mathcal{F}'' \in D^{\mathrm{b}}_{G_{\mathbf{d}}}(\mathrm{Rep}_{\mathbf{d}}(Q))$ by

$$\mathcal{F}' \diamond \mathcal{F}'' = R\varpi''_* \tilde{\mathcal{F}} \qquad \text{where } \tilde{\mathcal{F}} \in D^{\mathrm{b}}_{G_{\mathbf{d}}}(\mathbf{E}'') \text{ is the unique object such that } \varpi'^\sharp \tilde{\mathcal{F}} = \varpi^\sharp(\mathcal{F}' \boxtimes \mathcal{F}'').$$

From the earlier dimension calculations, one finds that

$$\mathcal{F}' \diamond \mathcal{F}'' \cong (\mathcal{F}' * \mathcal{F}'')[\langle\!\langle \mathbf{d}', \mathbf{d}'' \rangle\!\rangle].$$

There is a version $\mathcal{K}_{\mathbf{d}}$ of the Grothendieck group of $D^{\mathrm{b}}_{G_{\mathbf{d}}}(\mathrm{Rep}_{\mathbf{d}}(Q))$ that is a module over $\mathbb{Z}[v, v^{-1}]$, with the property that $[\mathrm{IC}(\mathcal{O}_{\mathbf{n}})[n]] = v^{-n}[\mathrm{IC}(\mathcal{O}_{\mathbf{n}})]$. The equation above implies that if $\mathcal{O}_{\mathbf{n}'} \subset \mathrm{Rep}_{\mathbf{d}'}(Q)$ and $\mathcal{O}_{\mathbf{n}''} \subset \mathrm{Rep}_{\mathbf{d}''}(Q)$, then in $\mathcal{K}$,

$$[\mathrm{IC}(\mathcal{O}_{\mathbf{n}'}) \diamond \mathrm{IC}(\mathcal{O}_{\mathbf{n}''})] = v^{-\langle\!\langle \mathbf{d}', \mathbf{d}'' \rangle\!\rangle}[\mathrm{IC}(\mathcal{O}_{\mathbf{n}'}) * \mathrm{IC}(\mathcal{O}_{\mathbf{n}''})].$$

Using nontrivial facts about mixed sheaves on $\mathrm{Rep}_{\mathbf{d}}(Q)$, one can show that in $\mathcal{K}$,

$$[\mathrm{IC}(\mathcal{O}_{\mathbf{n}})] = v^{-\dim \mathcal{O}_{\mathbf{n}}}[\Xi_{\mathbf{n}}] + \sum_{\substack{\mathbf{n}' : \mathcal{O}_{\mathbf{n}'} \subset \overline{\mathcal{O}_{\mathbf{n}}};\ \mathbf{n}' \neq \mathbf{n} \\ i < 0}} v^{-\dim \mathcal{O}_{\mathbf{n}'} + i} \operatorname{rk} \mathcal{H}^{-\dim \mathcal{O}_{\mathbf{n}'} + i}(\mathrm{IC}(\mathcal{O}_{\mathbf{n}})|_{\mathcal{O}_{\mathbf{n}'}})[\Xi_{\mathbf{n}'}] \tag{4}$$

The restriction $i < 0$ is written here for emphasis, but it is not strictly necessary: the terms in the sum vanish for $i \geq 0$ anyway, by one of the defining properties of IC objects.

Theorem (Lusztig). There is an isomorphism of $\mathbb{Z}[v, v^{-1}]$-modules

$$\mathcal{K} := \bigoplus_{\mathbf{d} : \Delta \to \mathbb{Z}_{\geq 0}} \mathcal{K}_{\mathbf{d}} \xrightarrow{\sim} U^+_{\mathrm{L},v}(\mathfrak{g}) \tag{5}$$

such that we have the following behavior of products and objects:

$$\begin{array}{ccccccccc}
\mathcal{K} & \xrightarrow{\ \sim\ } & U^+_{\mathrm{L},v}(\mathfrak{g}) & \xrightarrow{v \mapsto \sqrt{q}} & U^+_{\sqrt{q}}(\mathfrak{g}) & \xrightarrow[\sim]{\text{Ringel, } \diamond} & \mathcal{H}_{\mathrm{Rep}(Q)} & \xrightarrow{\ \sim\ } & F(Q) \\
\diamond & \longleftrightarrow & \text{mult.} & \longrightarrow & \text{mult.} & \longleftrightarrow & \diamond & & \\
* & \longrightarrow & & & & & * & \longleftrightarrow & * \\
[\Xi_{\mathbf{n}}] & \longleftrightarrow & \diamond\text{-lift of } [M(\mathbf{n})] & & \longrightarrow & & [M(\mathbf{n})] & \longleftrightarrow & \chi_{\mathbf{n}}
\end{array}$$

Here, the "$\diamond$-lifts of $[M(\mathbf{n})]$" are the elements whose existence you proved in Problem Set 2, Exercise 5. (They are called "$\diamond$-lifts" because we are using the $\diamond$ version of Ringel's isomorphism; the $\cdot$ version would give different elements.)

The formula (4) highlights the importance of the elements $v^{-\dim \mathcal{O}_n}[\Xi_{\mathbf{n}}]$. Let us introduce the elements

$$\mathsf{E}_{\mathbf{n}} := v^{-\dim \mathcal{O}_{\mathbf{n}}}(\diamond\text{-lift of } [M(\mathbf{n})]) = \text{image of } v^{-\dim \mathcal{O}_{\mathbf{n}}}[\Xi_{\mathbf{n}}] \text{ under Eq. (5).}$$

The elements $\{\mathsf{E}_{\mathbf{n}}\}$ form the *PBW-type basis* of $U^+_{\mathrm{L},v}(\mathfrak{g})$.

Canonical basis. Let $\mathsf{B}_{\mathbf{n}}$ be the image of $[\mathrm{IC}(\mathcal{O}_{\mathbf{n}})]$ under (5). These elements form a new basis of $U^+_{\mathrm{L},v}(\mathfrak{g})$, called the *canonical basis*. This basis has two special properties: (i) *Positivity*: the Decomposition Theorem implies that

$$\mathsf{B}_{\mathbf{n}}\mathsf{B}_{\mathbf{n}'} = \sum m_{\mathbf{n}''}\mathsf{B}_{\mathbf{n}''} \qquad \text{where each } m_{\mathbf{n}''} \in \mathbb{Z}[v, v^{-1}] \text{ is is } \overline{}\text{-stable and } \textit{has nonnegative coefficients}.$$

(ii) *Independence of Q:* If Q' is another quiver with the same underlying undirected graph as Q, there is an equivalence of derived categories $D^{\mathrm{b}}_{G_{\mathbf{d}}}(\mathrm{Rep}_{\mathbf{d}}(Q)) \cong D^{\mathrm{b}}_{G_{\mathbf{d}}}(\mathrm{Rep}_{\mathbf{d}}(Q'))$ given by a kind of Fourier transform. This equivalence commutes with $\diamond$-convolution, and it sends IC's to IC's (although it does *not* preserve the labels of orbits). See [1, §13].

The basis elements $\mathsf{B}_{\mathbf{n}}$ can be computed explicitly, using an algebraic analogues of the characterization of the $\mathrm{IC}(\mathcal{O}_{\mathbf{n}})$'s. First, an analogue of Verdier duality: define an involution

$$\overline{} : U^+_{\mathrm{L},v}(\mathfrak{g}) \to U^+_{\mathrm{L},v}(\mathfrak{g}) \qquad \text{by} \qquad \overline{E_\alpha} = E_\alpha \ (\alpha \in \Delta), \qquad \overline{v} = v^{-1}.$$

Let us now translate (4) across the isomorphism (5). For each $\mathbf{n} : \Phi^+ \to \mathbb{Z}_{\geq 0}$, $\mathsf{B}_{\mathbf{n}}$ is the unique element of $U^+_{\mathrm{L},v}(\mathfrak{g})$ such that

$$\mathsf{B}_{\mathbf{n}} = \mathsf{E}_{\mathbf{n}} + \sum_{\substack{\mathbf{m}:\Phi^+\to\mathbb{Z}_{\geq 0} \\ \mathcal{O}_{\mathbf{m}} \subset \overline{\mathcal{O}_{\mathbf{n}}},\ \mathbf{m}\neq\mathbf{n}}} P_{\mathbf{n},\mathbf{m}}\mathsf{E}_{\mathbf{m}} \quad \text{with } P_{\mathbf{n},\mathbf{m}} \in v^{-1}\mathbb{Z}[v^{-1}], \qquad \text{and} \qquad \overline{\mathsf{B}_{\mathbf{n}}} = \mathsf{B}_{\mathbf{n}}.$$

If we write them in the $\{[M(\mathbf{n})]\}$ basis (or, to be precise, in the $\{\diamond\text{-lift of } [M(\mathbf{n})]\}$ basis) instead, we see that $\mathsf{B}_{\mathbf{n}}$ is uniquely characterized by the properties that

$$\mathsf{B}_{\mathbf{n}} = v^{-\dim \mathcal{O}_{\mathbf{n}}}[M(\mathbf{n})] + \sum_{\substack{\mathbf{m}:\Phi^+\to\mathbb{Z}_{\geq 0} \\ \mathcal{O}_{\mathbf{m}} \subset \overline{\mathcal{O}_{\mathbf{n}}},\ \mathbf{m}\neq\mathbf{n}}} P_{\mathbf{n},\mathbf{m}}[M(\mathbf{m})] \quad \text{with } P_{\mathbf{n},\mathbf{m}} \in v^{-\dim \mathcal{O}_{\mathbf{m}}-1}\mathbb{Z}[v^{-1}], \qquad \text{and} \qquad \overline{\mathsf{B}_{\mathbf{n}}} = \mathsf{B}_{\mathbf{n}}.$$

Aside. If we use the $\cdot$ version of Ringel's theorem to lift the $[M(\mathbf{n})]$'s instead, the setup looks like this:

$$\begin{array}{ccccccc}
\mathcal{K} & \xrightarrow{\sim} & U^+_{\mathrm{L},v}(\mathfrak{g}) & \xrightarrow{v\mapsto\sqrt{q}} & U^+_{\sqrt{q}}(\mathfrak{g}) & \xrightarrow[\sim]{\text{Ringel, }\cdot} & \mathcal{H}_{\mathrm{Rep}(Q)} \\
\diamond & \longleftrightarrow & \text{mult.} & \longrightarrow & \text{mult.} & \longleftrightarrow & \cdot \\
[\Xi_{\mathbf{n}}] & \longleftrightarrow & v^{\dim \mathcal{O}_{\mathbf{n}} + \dim \mathrm{End}(M(\mathbf{n})) - \dim M(\mathbf{n})}(\cdot\text{-lift of } [M(\mathbf{n})]) & & \longrightarrow & & [M(\mathbf{n})] \\
v^{-\dim \mathcal{O}_{\mathbf{n}}}[\Xi_{\mathbf{n}}] & \longleftrightarrow & \mathsf{E}_{\mathbf{n}} = v^{\dim \mathrm{End}(M(\mathbf{n})) - \dim M(\mathbf{n})}(\cdot\text{-lift of } [M(\mathbf{n})]) & & & &
\end{array}$$

This version matches Theorems 4.11 and 4.12 from Rouquier's notes. Note that both the PBW basis $\{\mathsf{E}_{\mathbf{n}}\}$ and the canonical basis $\{\mathsf{B}_{\mathbf{n}}\}$ are the same as above, but the $\{\text{lift of } [M(\mathbf{n})]\}$ basis is different.

Many sources seem to use the $\cdot$ version, which looks really unnatural to me from the geometric perspective. Lusztig doesn't use either $\cdot$ or $\diamond$; instead, he uses $*$, with a slightly different version of Ringel's theorem that incorporates the elements $K_\alpha^{\pm 1} \in U_{\mathrm{L},v}(\mathfrak{g})$.

Stability for Quivers

$Q=(I,E)$, $\theta\in\mathbb{Z}^I$, $\mathbf{v}\in\mathbb{Z}^I_{\geq 0}$

Assume: $\theta\cdot\mathbf{v}=0$

$V\in\mathrm{Rep}(Q,\mathbf{v})$ **semistable:** $\forall$ proper subreps $N\subset V$, $\mathrm{slope}_\theta(N)\leq\mathrm{slope}_\theta(V)$

stable: $<$ instead of $\leq$

Thm. Semistable reps of fixed slope form an ab. cat. Stable reps are simple objs.

S-equivalent: same stable composition factors in this abelian cat.

GIT

$G\curvearrowright X$, $\chi: G\to\mathbb{C}^\times$

semistable: $\exists\ \chi^n$-semi-invt. f: $f(x)\neq 0$

stable: fin. stabilizer, orbit closed in X^{ss}

S-equivalent: orbit closures meet in X^{ss}

GIT quot.: $X/\!/_\chi G\leftrightarrow S$-equiv. classes

Maps: $X^{ss}\to X/\!/_\chi G\to X/\!/G$

King's Theorem

$\theta\in\mathbb{Z}^I$ s.t. $\theta\cdot\mathbf{v}=0\rightsquigarrow\chi_\theta:\bar{G}_\mathbf{v}\to\mathbb{C}^\times$

The two kinds of (semi)stability coincide for $\bar{G}_\mathbf{v}\curvearrowright\mathrm{Rep}(Q,\mathbf{v})$ and $\mathrm{Rep}(Q,\mathbf{v})/\!/_{\chi_\theta}\bar{G}_\mathbf{v}$.

Framings

$Q^\heartsuit$, $\mathbf{v},\mathbf{w}\in\mathbb{Z}^I_{\geq 0}$. $G_\mathbf{v}\curvearrowright\mathrm{Rep}(Q^\heartsuit,\mathbf{v},\mathbf{w})$

Or: Fix $\mathbf{w}\in\mathbb{Z}^I_{\geq 0}$. $G_\mathbf{v}\curvearrowright\mathrm{Rep}(Q^\mathbf{w},\hat{\mathbf{v}})$

Isomorphic varieties. Both fine for GIT.

King's Thm for Framings

$\theta\in\mathbb{Z}^I$. No assumption on $\theta\cdot\mathbf{v}$.

But $\hat\theta\in\mathbb{Z}^{I\cup\{\infty\}}$ always has $\hat\theta\cdot\hat{\mathbf{v}}=0$.

Must use $\mathrm{Rep}(Q^\mathbf{w},\hat{\mathbf{v}})$ for algebraic notion.

Alg. $\hat\theta$-(semi-)stability for $\mathrm{Rep}(Q^\mathbf{w},\hat{\mathbf{v}})$ coincides with GIT (semi)stability for $G_\mathbf{v}\curvearrowright\mathrm{Rep}(Q^\heartsuit,\mathbf{v},\mathbf{w})$ and $\mathrm{Rep}(Q^\heartsuit,\mathbf{v},\mathbf{w})/\!/_{\chi_\theta}G_\mathbf{v}$

Nakajima Quiver Varieties

Q, $\mathbf{v},\mathbf{w}\in\mathbb{Z}^I$, $\theta\in\mathbb{Z}^I$, $\lambda\in\mathbb{C}^I$.

No assumptions on $\theta\cdot\mathbf{v}$ or $\lambda\cdot\mathbf{v}$

μ: $T^*\mathrm{Rep}(Q^\heartsuit,\mathbf{v},\mathbf{w})=\mathrm{Rep}(\overline{Q^\heartsuit},\mathbf{v},\mathbf{w})\to\mathfrak{g}_\mathbf{v}$

$(\mathbf{x},\mathbf{y},\mathbf{i},\mathbf{j})\mapsto[\mathbf{x},\mathbf{y}]+\mathbf{i}\circ\mathbf{j}$

Defn.: $\mathcal{M}_{\lambda,\theta}(\mathbf{v},\mathbf{w}):=\mu^{-1}(\lambda)/\!/_{\chi_\theta}G_\mathbf{v}$

Stability: use $\mathrm{Rep}(\overline{Q^\mathbf{w}},\hat{\mathbf{v}})$, $\Pi_{\hat\lambda}(Q^\mathbf{w})$-mod.

$V\in\mathrm{Rep}(\overline{Q^\heartsuit},\mathbf{v},\mathbf{w})$ **semistable:** all proper $\overline{Q}$-subreps $N\subset V$ satisfy:

$N_i\subset\ker\mathbf{j}_i\ \forall i\in I\Rightarrow\theta\cdot\underline{\dim}\,N\leq 0$,

$N_i\supset\mathrm{im}\,\mathbf{i}_i\ \forall i\in I\Rightarrow\theta\cdot\underline{\dim}\,N\leq\theta\cdot\mathbf{v}$.

stable: $<$ instead of $\leq$

Symplectic Geometry

$G\curvearrowright X$ induces $G\curvearrowright T^*X$.

Moment map: $\mu: T^*X\to\mathfrak{g}^*$. Look at GIT quotients $\mu^{-1}(\lambda)/\!/_\chi G$, esp. $\lambda=0$.

$\mu^{-1}(0)=\bigcup(\text{conormal bdles of orbits in } X)$

Symplectic Geometry for Quivers

$T^*\mathrm{Rep}(Q,\mathbf{v})=\mathrm{Rep}(\overline{Q},\mathbf{v})$. $\mathfrak{g}_\mathbf{v}^*\cong\mathfrak{g}_\mathbf{v}$.

$\mu:\mathrm{Rep}(\overline{Q},\mathbf{v})\to\mathfrak{g}_\mathbf{v}$: $(\mathbf{x},\mathbf{y})\mapsto\mathbf{x}\circ\mathbf{y}-\mathbf{y}\circ\mathbf{x}$

$\lambda\in\mathbb{C}^I$, regarded as elt. of $\mathfrak{g}_\mathbf{v}$

Assume: $\lambda\cdot\mathbf{v}=0$

Preprojective algebra: $\Pi_\lambda=\Pi_\lambda(Q)$.

$\{\Pi_\lambda\text{-mod. of dim. vec. }\mathbf{v}\}\leftrightarrow\mu^{-1}(\lambda)$

King's Thm for Doubled Quivers

$\theta\in\mathbb{Z}^I$, $\lambda\in\mathbb{C}^I$ s.t $\theta\cdot\mathbf{v}=\lambda\cdot\mathbf{v}=0$.

Algebraic (semi)stability for Π_λ-modules coincides with GIT (semi)stability for $\bar{G}_\mathbf{v}\curvearrowright\mu^{-1}(\lambda)$ and $\mu^{-1}(\lambda)/\!/_{\chi_\theta}\bar{G}_\mathbf{v}$.

Pop Quiz

1. What are the definitions of $Q^\mathbf{w}$, $\hat{\mathbf{v}}$, and $\hat\theta$?
2. Some boxes have assumptions like $\theta\cdot\mathbf{v}=0$ or $\lambda\cdot\mathbf{v}=0$, and others don't. Explain these assumptions.
3. Some boxes use the action of $\bar{G}_\mathbf{v}$, and others use $G_\mathbf{v}$. Explain why.

References

[1] G. Lusztig, *Canonical bases arising from quantized enveloping algebras*, J. Amer. Math. Soc. **3** (1990), 447–498.