# Problem Set 2

*Due: February 13, 2015*

1. Let $k = \mathbb{F}_q$ be the finite field with $q$ elements. Let $n \geq m \geq 0$. Prove that the number of $m$-dimensional subspaces of $k^n$ is the $q$-binomial coefficient $\binom{n}{m}_q$. (Recall that this is defined to be $\frac{[n]_q!}{[m]_q![n-m]_q!}$, where $[n]_q! = [n]_q[n-1]_q \cdots [1]_q$, and where $[n]_q = \frac{q^n-1}{q-1} = q^{n-1} + q^{n-2} + \cdots + q + 1$.)

   It might be useful to first study the number of complete flags in $k^n$. A *complete flag* is a sequence of subspaces $0 = V_0 \subset V_1 \subset \cdots \subset V_n = k^n$ where $\dim V_i = i$. Show that $k^n$ contains $[n]_q^!$ complete flags.

2. Let $V$ be a real inner product space. For a nonzero vector $\alpha \in V$, let $\alpha^\vee = \frac{2\alpha}{\langle \alpha, \alpha \rangle}$. The map

$$s_\alpha : V \to V \qquad \text{given by} \qquad s_\alpha(v) = v - \langle \alpha^\vee, v \rangle \alpha$$

   is called *reflection across the hyperplane perpendicular to* $\alpha$. This name is justified by the fact that $s_\alpha(\alpha) = -\alpha$, and that $s_\alpha(v) = v$ if $\langle \alpha, v \rangle = 0$.

   Recall that an *abstract root system* is a finite set $\Phi$ of nonzero vectors in a real inner product space $V$ satisfying the following axioms:

   (a) $\Phi$ spans $V$.
   (b) For all $\alpha \in \Phi$, we have $\mathbb{R}\alpha \cap \Phi = \{\alpha, -\alpha\}$.
   (c) For all $\alpha, \beta \in \Phi$, we have $s_\alpha(\beta) \in \Phi$.
   (d) For all $\alpha, \beta \in \Phi$, we have $\langle \alpha^\vee, \beta \rangle \in \mathbb{Z}$.

   Here is an example. Let $\varepsilon_1, \ldots, \varepsilon_n$ be the standard basis vectors for $\mathbb{R}^n$, equipped with its usual Euclidean inner product. Let $V \subset \mathbb{R}^n$ be the space of vectors $\sum v_i \varepsilon_i$ such that $\sum v_i = 0$.

   (a) Let $\Phi = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i, j \leq n,\ i \neq j\}$. Prove that $\Phi$ is an abstract root system.
   (b) Let $\Phi^+ = \{\varepsilon_i - \varepsilon_j \mid i < j\}$, and call these vectors *positive roots*. Determine the set $\Delta$ of simple roots (i.e., positive roots that are not the sum of two other positive roots), and draw the Dynkin diagram for this root system. (*Answer:* It is type $A_{n-1}$.)
   (c) Show that the two notions of *positive root* (element of $\Phi^+$, or vector in $\mathbb{Z}_{\geq 0}\Delta$ on which the quadratic form $q$ takes value 1) coincide.

3. Recall that $U_v^+(\mathfrak{g})$ admits a grading by $\mathbb{Z}^{|\Delta|} \cong \{\mathbf{d} : \Delta \to \mathbb{Z}\}$, by setting $\deg E_\alpha = \underline{\dim}(\alpha)$ for $\alpha \in \Delta$. This induces a grading on any specialization $U_q^+(\mathfrak{g})$. Prove that for any $q \in \mathbb{C}^\times$, we have

$$\dim U_q^+(\mathfrak{g})^{\mathbf{d}} = |\{\mathbf{n} : \Phi^+ \to \mathbb{Z}_{\geq 0} \mid \underline{\dim}(\mathbf{n}) = \mathbf{d}\}|.$$

   (*Hint:* You will have to use the nontrivial fact that $U_1^+(\mathfrak{g})$ is isomorphic to the universal enveloping algebra $U(\mathfrak{n})$, where $\mathfrak{n} = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$. First prove the dimension calculation in the case $q = 1$, using the Poincaré–Birkhoff–Witt theorem for $U(\mathfrak{n})$. Then deduce it for general $q$ using the fact that $U_v^+(\mathfrak{g})$ is free over $\mathbb{C}[v, v^{-1}]$.)

4. Show that the modified product on the Hall algebra (shown below) is associative.

$$[M] \cdot [N] = t^{\langle M, N \rangle}[M] * [N]$$

5. Via Ringel's isomorphism $\phi : U_{\sqrt{q}}^+(\mathfrak{g}) \xrightarrow{\sim} \mathcal{H}_{\mathrm{Rep}(Q)}$, we can equip $U_{\sqrt{q}}^+(\mathfrak{g})$ with the basis $\{\phi^{-1}([M(\mathbf{n})]) \mid \mathbf{n} : \Phi^+ \to \mathbb{Z}_{\geq 0}\}$. Prove that this basis is "independent of $q$": i.e., that it lifts to a $\mathbb{Z}[v, v^{-1}]$ basis of $U_{\mathrm{L},v}^+(\mathfrak{g})$. (*Hint:* Use an induction argument like the one we used in class to show that $\phi$ is surjective.)