Problem Set 3

Due: March 13, 2015

1. (Optional) Let $X = \bigsqcup_{s \in J} X_s$ be a stratified complex variety, with dense stratum X_{s_0}. Let $f : Y \to X$ be a surjective proper map, with Y smooth. Assume that f is a *small map*, which means that
$$\dim f^{-1}(x) \le \tfrac{1}{2}(\dim X_{s_0} - \dim X_t) \qquad \text{for } x \in X_t, \text{ with equality only if } t = s_0.$$
Show that $Rf_* \underline{\mathbb{C}}_Y[\dim Y] \cong \mathrm{IC}(X_{s_0})$. (This is easy if you are used to working with the tools of sheaf theory, such as the "proper base change theorem.")

2. Consider the quiver $Q = \bullet$. The Hall algebra $\mathcal{H}_{\mathrm{Rep}(Q)}$ has a basis $\{[k^n]\}_{n \in Z_{\ge 0}}$. Show that when you lift this basis to a basis of $U^+_{\mathrm{L},v}(\mathfrak{sl}_2)$ as in Problem Set 2, Exercise 5, the basis element $[k^n]$ lifts to $E_\alpha^{(n)}$.

3. Fill in as many rows of the in-class worksheet as you can. Extra brownie points if you can give a general formula for the $[M(\mathbf{n})]$'s.

4. Determine the canonical basis for $U^+_{\mathrm{L},v}(\mathfrak{sl}_3)$. In more detail, fix a dimension vector $\mathbf{d} : \Delta \to \mathbb{Z}_{\ge 0}$, say
$$\mathbf{d}(\alpha) = p, \qquad \mathbf{d}(\beta) = q.$$
Then the possible $\mathbf{n} : \Phi^+ \to \mathbb{Z}_{\ge 0}$ with $\underline{\dim}(\mathbf{n}) = \mathbf{d}$ are of the form
$$\mathbf{n}(\beta) = p - r, \qquad \mathbf{n}(\alpha + \beta) = r, \qquad \mathbf{n}(\alpha) = q - r, \qquad \text{for some } r,\ 0 \le r \le \min\{p, q\}.$$
Show that
$$B_{\mathbf{n}} = \begin{cases} E_\alpha^{(r)} E_\beta^{(q)} E_\alpha^{(p-r)} & \text{if } p \le q, \\ E_\beta^{(q-r)} E_\alpha^{(p)} E_\beta^{(r)} & \text{if } p \ge q. \end{cases}$$
(The two formulas coincide with $p = q$.) There are two ways to solve this:

Algebraically: do this with quantum group and Hall algebra calculations, like on the in-class worksheet. I haven't done this myself, and I do not know how much work it is.

Geometrically: this approach uses more machinery, but it is more conceptual.

   (a) Assume for now that $p \le q$. Let
   $$\mathbf{E} = \{(\phi, M_1 \subset M_2 \subset M(\phi)) \mid \phi \in \mathrm{Rep}_{\mathbf{d}}(Q),\ M_1 \cong V_\alpha^{\oplus(p-r)},\ M_2/M_1 \cong V_\beta^{\oplus q},\ M(\phi)/M_2 \cong V_\alpha^{\oplus r}\}.$$
   Let $\pi : \mathbf{E} \to \mathrm{Rep}_{\mathbf{d}}(Q)$ be the obvious map $(\phi, M_1 \subset M_2 \subset M(\phi)) \mapsto \phi$. Show, by unraveling the definition of convolution of sheaves, that $E_\alpha^{(r)} E_\beta^{(q)} E_\alpha^{(p-r)}$ is the class of $R\pi_* \underline{\mathbb{C}}_{\mathbf{E}}[\dim \mathbf{E}]$.

   (b) Show that $\mathbf{E}$ is a smooth variety.

   (c) Determine the closure $\overline{\mathcal{O}_{\mathbf{n}}}$. Specifically, let $\mathcal{O}_{\mathbf{m}} \subset \mathrm{Rep}_{\mathbf{d}}(Q)$ be another orbit, say with $\mathbf{m}(\beta) = p - s$, $\mathbf{m}(\alpha + \beta) = s$, and $\mathbf{m}(\alpha) = q - s$. Then $\mathcal{O}_{\mathbf{m}} \subset \mathcal{O}_{\mathbf{n}}$ if and only if $s \le r$.

   (d) Let $\phi \in \mathrm{Rep}_{\mathbf{d}}(Q)$, and compute $\pi^{-1}(\phi)$. The answer depends on the orbit $\mathcal{O}_{\mathbf{m}}$ to which ϕ belongs. Suppose $\mathbf{m}(\beta) = p - s$, $\mathbf{m}(\alpha + \beta) = s$, and $\mathbf{m}(\alpha) = q - s$. Then
   $$\pi^{-1}(\phi) = \varnothing \quad \text{if } s > r, \qquad \pi^{-1}(\phi) \cong \mathrm{Gr}(p - r, p - s) \quad \text{if } s \le r.$$
   In particular, the image of π is exactly $\overline{\mathcal{O}_{\mathbf{n}}}$.

   (e) Show that $\dim \mathcal{O}_{\mathbf{m}} = s(p + q - s)$. (*Hint*: Identify $\mathcal{O}_{\mathbf{m}}$ with the space of linear maps $k^p \to k^q$ of rank s.) Then show that $\pi : \mathbf{E} \to \overline{\mathcal{O}_{\mathbf{n}}}$ is a *small map* as in Exercise 1. That is, if $\phi \in \mathcal{O}_{\mathbf{m}} \subset \overline{\mathcal{O}_{\mathbf{n}}}$, then $\dim \pi^{-1}(\phi) \le \frac{1}{2}(\dim \mathcal{O}_{\mathbf{n}} - \dim \mathcal{O}_{\mathbf{m}})$, with equality if and only if $\mathcal{O}_{\mathbf{n}} = \mathcal{O}_{\mathbf{m}}$, i.e., if $s = r$.

Combining these results with Exercise 1, one finds that $E_\alpha^{(r)} E_\beta^{(q)} E_\alpha^{(p-r)} = [R\pi_* \underline{\mathbb{C}}_{\mathbf{E}}[\dim \mathbf{E}]] = [\mathrm{IC}(\mathcal{O}_{\mathbf{n}})] = B_{\mathbf{n}}$, as desired. Now produce a similar argument for the case $p \ge q$.