Problem Set 4

Due: May 8, 2015

For this problem set, let n be a positive integer, and let Q be the following quiver with $n-1$ vertices:

$$\bullet \xrightarrow[x_1]{} \bullet \xrightarrow[x_2]{} \bullet \xrightarrow[x_3]{} \cdots \xrightarrow[x_{n-2}]{} \bullet$$

Let

$$\mathbf{v} = (1, 2, \ldots, n-1), \qquad \mathbf{w} = (0, 0, \ldots, 0, n), \qquad \theta = \theta^+ = (1, 1, \ldots, 1).$$

Since most coordinates of $\mathbf{w}$ are 0, one can ignore most of the shadow vertices. Thus, elements of $\mathrm{Rep}(Q^\heartsuit, \mathbf{v}, \mathbf{w})$ and $\mathrm{Rep}(\overline{Q^\heartsuit}, \mathbf{v}, \mathbf{w})$, respectively, look like

$$\begin{tikzcd} & & & & \mathbb{C}^n \\ \mathbb{C} \arrow[r, "x_1"'] & \mathbb{C}^2 \arrow[r, "x_2"'] & \mathbb{C}^3 \arrow[r, "x_3"'] & \cdots \arrow[r, "x_{n-2}"'] & \mathbb{C}^{n-1} \arrow[u, "\mathbf{j}"'] \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} & & & & \mathbb{C}^n \arrow[d, shift right, "\mathbf{i}"'] \\ \mathbb{C} \arrow[r, shift right, "x_1"'] & \mathbb{C}^2 \arrow[l, shift right, "x_1^*"'] \arrow[r, shift right, "x_2"'] & \mathbb{C}^3 \arrow[l, shift right, "x_2^*"'] \arrow[r, shift right, "x_3"'] & \cdots \arrow[l, shift right, "x_3^*"'] \arrow[r, shift right, "x_{n-2}"'] & \mathbb{C}^{n-1} \arrow[l, shift right, "x_{n-2}^*"'] \arrow[u, shift right, "\mathbf{j}"'] \end{tikzcd}$$

1. Show that an element of $\mathrm{Rep}(Q^\heartsuit, \mathbf{v}, \mathbf{w})$ is θ-semistable if and only if all the x_k and $\mathbf{j}$ are injective maps. Show also that every such point is in fact θ-stable. Then show that
$$\mathrm{Rep}(Q^\heartsuit, \mathbf{v}, \mathbf{w})/\!/_{\chi_\theta} G_{\mathbf{v}} \cong \mathcal{F}_n,$$
where $\mathcal{F}_n$ is the variety of complete flags in $\mathbb{C}^n$. Describe the natural map $\mathrm{Rep}(Q^\heartsuit, \mathbf{v}, \mathbf{w})^{ss} \to \mathrm{Rep}(Q^\heartsuit, \mathbf{v}, \mathbf{w})/\!/_{\chi_\theta} G_{\mathbf{v}}$ explicitly. (We did the $n = 1$ and $n = 2$ cases of this in class.)

2. An element of $\mathcal{F}_n$ is a sequence $F = (0 \subset F_1 \subset F_2 \subset \cdots \subset F_n = \mathbb{C}^n)$ of subspaces of $\mathbb{C}^n$ such that $\dim F_i = i$. Consider the space
$$X = \{(F, x) \in \mathcal{F}_n \times \mathfrak{gl}_n \mid x(F_i) \subset F_{i-1} \text{ for all } i,\ 1 \le i \le n\}.$$
It turns out that $X \cong T^*\mathcal{F}_n$. This problem asks you to prove something slightly weaker: give a natural isomorphism between the fibers of the projection map $X \to \mathcal{F}_n$ and the cotangent spaces of $\mathcal{F}_n$. *Hint:* Fix a flag F, and let $B \subset GL_n$ be the subgroup of matrices that preserve F. With a suitable choice of basis, B can be identified with the group of upper-triangular matrices. Show that $\mathcal{F}_n \cong GL_n/B$. Therefore, the tangent space to $\mathcal{F}_n$ at F can be identified with $\mathfrak{gl}_n/\mathfrak{b}$. Then show that $(\mathfrak{gl}_n/\mathfrak{b})^* \cong \{x \in \mathfrak{gl}_n \mid x(F_i) \subset F_{i-1}\}$.

3. Show that an element of $\mu^{-1}(0) \subset \mathrm{Rep}(\overline{Q^\heartsuit}, \mathbf{v}, \mathbf{w})$ is θ-semistable if and only if all the x_k and $\mathbf{j}$ are injective maps. Show also that every such point is in fact θ-stable. Then show that
$$\mathcal{M}_{0,\theta}(\mathbf{v}, \mathbf{w}) = \mu^{-1}(0)/\!/_{\chi_\theta} G_{\mathbf{v}} \cong T^*\mathcal{F}_n.$$
(*Hint:* Here is how to define a map $\mathcal{M}_{0,\theta}(\mathbf{v}, \mathbf{w}) \to X$, with X as in Problem 2. Define the flag F as in Problem 1, and define $x \in \mathfrak{gl_n}$ to be the map $\mathbf{j} \circ \mathbf{i}$.)

4. Let $\mathcal{N}$ be the variety of $n \times n$ nilpotent matrices in $\mathfrak{gl}_n$. It is known that
$$\mu^{-1}(0)/\!/G_{\mathbf{v}} \cong \mathcal{N}.$$
Describe the map $\mathcal{M}_{0,\theta}(\mathbf{v}.\mathbf{w}) \to \mathcal{N}$ explicitly (the *Springer resolution*). Prove the isomorphism above for $n = 1, 2$. The proof for general n requires additional background in invariant theory, but you should at least show how it follows from the following claim: $\mathcal{N} \cong Y/\!/GL_{n-1}$, where Y is the variety of diagrams
$$\begin{tikzcd} \mathbb{C}^{n-1} \arrow[loop left, "y"] \arrow[r, shift right, "\mathbf{j}"'] & \mathbb{C}^n \arrow[l, shift right, "\mathbf{i}"'] \end{tikzcd} \qquad \text{such that } y \text{ is nilpotent and } y = \mathbf{i} \circ \mathbf{j}.$$