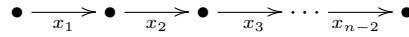


Problem Set 4

Due: May 8, 2015

For this problem set, let n be a positive integer, and let Q be the following quiver with $n - 1$ vertices:



Let

$$\mathbf{v} = (1, 2, \dots, n - 1), \quad \mathbf{w} = (0, 0, \dots, 0, n), \quad \theta = \theta^+ = (1, 1, \dots, 1).$$

Since most coordinates of \mathbf{w} are 0, one can ignore most of the shadow vertices. Thus, elements of $\text{Rep}(Q^\heartsuit, \mathbf{v}, \mathbf{w})$ and $\text{Rep}(\overline{Q^\heartsuit}, \mathbf{v}, \mathbf{w})$, respectively, look like



1. Show that an element of $\text{Rep}(Q^\heartsuit, \mathbf{v}, \mathbf{w})$ is θ -semistable if and only if all the x_k and \mathbf{j} are injective maps. Show also that every such point is in fact θ -stable. Then show that

$$\text{Rep}(Q^\heartsuit, \mathbf{v}, \mathbf{w}) //_{\chi_\theta} G_{\mathbf{v}} \cong \mathcal{F}_n,$$

where \mathcal{F}_n is the variety of complete flags in \mathbb{C}^n . Describe the natural map $\text{Rep}(Q^\heartsuit, \mathbf{v}, \mathbf{w})^{ss} \rightarrow \text{Rep}(Q^\heartsuit, \mathbf{v}, \mathbf{w}) //_{\chi_\theta} G_{\mathbf{v}}$ explicitly. (We did the $n = 1$ and $n = 2$ cases of this in class.)

2. An element of \mathcal{F}_n is a sequence $F = (0 \subset F_1 \subset F_2 \subset \cdots \subset F_n = \mathbb{C}^n)$ of subspaces of \mathbb{C}^n such that $\dim F_i = i$. Consider the space

$$X = \{(F, x) \in \mathcal{F}_n \times \mathfrak{gl}_n \mid x(F_i) \subset F_{i-1} \text{ for all } i, 1 \leq i \leq n\}.$$

It turns out that $X \cong T^*\mathcal{F}_n$. This problem asks you to prove something slightly weaker: give a natural isomorphism between the fibers of the projection map $X \rightarrow \mathcal{F}_n$ and the cotangent spaces of \mathcal{F}_n . *Hint:* Fix a flag F , and let $B \subset GL_n$ be the subgroup of matrices that preserve F . With a suitable choice of basis, B can be identified with the group of upper-triangular matrices. Show that $\mathcal{F}_n \cong GL_n/B$. Therefore, the tangent space to \mathcal{F}_n at F can be identified with $\mathfrak{gl}_n/\mathfrak{b}$. Then show that $(\mathfrak{gl}_n/\mathfrak{b})^* \cong \{x \in \mathfrak{gl}_n \mid x(F_i) \subset F_{i-1}\}$.

3. Show that an element of $\mu^{-1}(0) \subset \text{Rep}(\overline{Q^\heartsuit}, \mathbf{v}, \mathbf{w})$ is θ -semistable if and only if all the x_k and \mathbf{j} are injective maps. Show also that every such point is in fact θ -stable. Then show that

$$\mathcal{M}_{0,\theta}(\mathbf{v}, \mathbf{w}) = \mu^{-1}(0) //_{\chi_\theta} G_{\mathbf{v}} \cong T^*\mathcal{F}_n.$$

(*Hint:* Here is how to define a map $\mathcal{M}_{0,\theta}(\mathbf{v}, \mathbf{w}) \rightarrow X$, with X as in Problem 2. Define the flag F as in Problem 1, and define $x \in \mathfrak{gl}_n$ to be the map $\mathbf{j} \circ \mathbf{i}$.)

4. Let \mathcal{N} be the variety of $n \times n$ nilpotent matrices in \mathfrak{gl}_n . It is known that

$$\mu^{-1}(0) // G_{\mathbf{v}} \cong \mathcal{N}.$$

Describe the map $\mathcal{M}_{0,\theta}(\mathbf{v}, \mathbf{w}) \rightarrow \mathcal{N}$ explicitly (the *Springer resolution*). Prove the isomorphism above for $n = 1, 2$. The proof for general n requires additional background in invariant theory, but you should at least show how it follows from the following claim: $\mathcal{N} \cong Y // GL_{n-1}$, where Y is the variety of diagrams

