Homework 3

Due: September 29, 2016

1. Prove Theorem 3.10 on page 24. You get brownie points if you do it without looking up the reference that the textbook gives. (*Hint*: It is easy to see that $C_G(T)^\circ \subset N_G(T)^\circ$. To prove that they are equal, imitate the end of the proof of the Lie–Kolchin theorem.)

2. Exercise 10.18.

3. Let $\mathrm{Gr}(r,n)$ be the Grassmannian of r-dimensional subspaces in k^n. Recall that this set can be identified with a subset of the projective space $\mathbb{P}(\bigwedge^r k^n)$ via the map

$$\mathrm{span}(v_1, \ldots, v_r) \mapsto \mathrm{span}(v_1 \wedge \cdots \wedge v_r).$$

   In class, I mentioned that one can show that the image is closed using the *Plücker equations*. This makes $\mathrm{Gr}(r,n)$ into a projective variety. This exercise outlines an alternative proof that $\mathrm{Gr}(r,n)$ is closed.

   Note that GL_n acts on the vector space $\bigwedge^r k^n$ and hence on $\mathbb{P}(\wedge^r k^n)$. Moreover, it acts transitively on the set of r-dimensional subspaces in k^n; in other words, $\mathrm{Gr}(r,n)$ is a single GL_n-orbit in $\mathbb{P}(\bigwedge^r k^n)$. (If you don't know why the previous sentence is true, you should prove it for yourself.) The idea is to show that this orbit has minimal dimension. Since orbits of minimal dimension are closed, we're done.

   (a) Let $T \subset \mathrm{GL}_n$ be the subgroup consisting of diagonal matrices. (This is a torus, of course.) Show that the T has exactly $\binom{n}{k}$ fixed points in $\mathbb{P}(\bigwedge^r k^n)$: namely, its fixed points are the lines spanned by r-fold wedge products of standard basis vectors in k^n. (Consider T-eigenvectors in $\bigwedge^r k^n$.) In particular, $\mathrm{Gr}(r,n)$ contains all the T-fixed points.

   (b) Let $X \subset \mathbb{P}(\bigwedge^r k^n)$ be a GL_n-orbit of minimal dimension. Use the Borel fixed point theorem to show that X contains a T-fixed point.

   (c) Conclude that $\mathrm{Gr}(r,n)$ is in fact the *unique* closed GL_n-orbit in $\mathbb{P}(\bigwedge^r k^n)$.

4. Exercise 10.20. The definition of *nilpotent group* is on page 6 of the textbook. This notion is close in spirit to that of *solvable group*; it has nothing to do with Jordan decomposition or unipotent groups (but see Corollary 2.10).

   The textbook gives a hint suggesting that you use Exercise 10.6, which I haven't previously assigned. You don't have to hand in Exercise 10.6, but you should figure it out and give a classmate a verbal pop quiz on it. As evidence that you did this, you should write down the name of the person whom you quizzed, and that of the person who quizzed you. They should be different!