Notes on Jordan Decomposition

Theorem 1 (Jordan Normal Form). *Let V be a finite-dimensional vector space. For any linear operator $g \in \mathrm{End}(V)$, there exists an element $x \in \mathrm{GL}(V)$ such that $x^{-1}gx$ is in "Jordan normal form," i.e., it is a block-diagonal matrix made up of Jordan blocks.*

Definition 2. Let V be a finite-dimensional vector space. An element $g \in \mathrm{End}(V)$ is called *semisimple* if it is conjugate to a diagonal matrix (equivalently, if V has a basis consisting of eigenvectors for g). It is called *nilpotent* if $g^n = 0$ for $n \gg 0$. It is called *unipotent* if $g - 1$ is nilpotent.

Theorem 3 (Additive Jordan Decomposition). *Let V be a finite-dimensional vector space. For any linear operator $g \in \mathrm{End}(V)$, there exists a unique pair of elements $s, n \in \mathrm{End}(V)$ such that:*

1. *s is semisimple and n is nilpotent;*
2. *s and n commute: $sn = ns$;*
3. *$g = s + n$.*

Theorem 4 (Multiplicative Jordan Decomposition). *Let V be a finite-dimensional vector space. For any linear operator $g \in \mathrm{GL}(V)$, there exists a unique pair of elements $s, u \in \mathrm{GL}(V)$ such that:*

1. *s is semisimple and u is unipotent;*
2. *s and u commute: $su = us$;*
3. *$g = su$.*

Definition 5. Let W be an infinite-dimensional vector space. A linear operator $\theta : W \to W$ is called *manageable* if every vector in W is contained in a finite-dimensional subspace that is preserved by θ.

A manageable operator $\theta : W \to W$ is called *semisimple*, resp. *unipotent*, if for every finite-dimensional θ-stable subspace $V \subset W$, the operator $\theta|_V : V \to V$ is semisimple, resp. unipotent. (The terms "semisimple" and "unipotent" are *not defined* for unmanageable operators.)

Theorem 6 (Infinite-Dimensional Jordan Decomposition). *Let W be a (possibly infinite-dimensional) vector space, and let $\theta : W \to W$ be a manageable invertible linear operator. Then there exists a unique pair of manageable invertible linear operators $\sigma, \upsilon : W \to W$ such that:*

1. *If $V \subset W$ is a finite-dimensional subspace preserved by θ, then σ and υ also preserve V. Moreover, $\sigma|_V$ is semisimple, and $\upsilon|_V$ is unipotent. (In particular, σ is semisimple and υ is unipotent.)*
2. *σ and υ commute: $\sigma\upsilon = \upsilon\sigma$.*
3. *$\theta = \sigma\upsilon$.*

Proposition 7. *Let G be an algebraic group. Any finite-dimensional subspace (and hence any vector) of $k[G]$ is contained in a finite-dimensional subspace that is preserved by all operators ρ_g ($g \in G$).*

In particular, the operators $\rho_g : k[G] \to k[G]$ are manageable.

Definition 8. Let G be an algebraic group. An element $g \in G$ is called *semisimple*, resp. *unipotent*, if ρ_g is a semisimple, resp. unipotent, operator in the sense of Definition 5.

Theorem 9. *For $\mathrm{GL}(V)$, Definitions 2 and 8 coincide.*

Theorem 10 (Jordan Decomposition for Algebraic Groups). *Let G be an algebraic group. For any $g \in G$, there exists a unique pair of elements $s, u \in G$ such that:*

1. *s is semisimple and u is unipotent (in the sense of Definition 8);*
2. *s and u commute: $su = us$;*
3. *$g = su$.*