Structure Theory of Algebraic Groups

Definition 1. A *torus* is an algebraic group that is isomorphic to $\mathbb{G}_{\mathrm{m}} \times \cdots \times \mathbb{G}_{\mathrm{m}}$.

Note that all elements of a torus are semisimple.

Theorem 2. *Every connected subgroup of a torus is a torus. More generally, any connected commutative group consisting only of semisimple elements is a torus.*

Definition 3. A *unipotent group* is a group all of whose elements are unipotent.

Theorem 4 (Unipotent Group Theorem)**.** *Every unipotent group is isomorphic to a subgroup of* $\left[\begin{smallmatrix} 1 & * & * \\ & \ddots & * \\ & & 1 \end{smallmatrix}\right]$.

Definition 5. A group G is said to be *solvable* if the sequence of subgroups $G' = [G,G]$, $G'' = [G', G']$, ... eventually reaches the trivial group.

Theorem 6 (Lie–Kolchin Theorem)**.** *Every solvable group is isomorphic to a subgroup of* $\left[\begin{smallmatrix} * & * & * \\ & \ddots & * \\ & & * \end{smallmatrix}\right]$.

Proposition 7. *Let G be a connected solvable group. The set of unipotent elements G_{u} forms a closed, connected, normal subgroup. Moreover, we have $[G,G] \subset G_{\mathrm{u}}$.*

In the setting of this proposition, G/G_{u} is connected, commutative, and consists only of semisimple elements, so it is a torus. Thus, for any connected solvable group, there is a short exact sequence of algebraic groups

$$1 \to G_{\mathrm{u}} \to G \to T \to 1 \qquad \text{where } T \text{ is a torus.}$$

Theorem 8 (did not prove)**.** *The short exact sequence above splits. In other words, every connected solvable subgroup G is isomorphic to a semidirect product $T \ltimes G_{\mathrm{u}}$.*

Such an isomorphism identifies T with a subgroup of G. Every subgroup of G that is isomorphic to T is in fact conjugate to T.

Definition 9. Let G be an algebraic group. A *Borel subgroup* is a connected, solvable closed subgroup that is not a proper subgroup of any other connected, solvable closed subgroup. In other words, a Borel subgroup is a maximal connected, solvable closed subgroup.

A *maximal torus* $T \subset G$ is a closed subgroup that is a torus, and that is not a proper subgroup of any other subgroup that is a torus.

So the previous (unproved) theorem says that all maximal tori in a connected solvable group are conjugate.

Theorem 10. *Let G be an algebraic group. (i) Any two Borel subgroups are conjugate. (ii) Any two maximal tori are conjugate.*

We proved part (i) in class. Part (ii) follows, using the unproved Theorem 8.

Theorem 11. *Borel subgroups are as non-normal as possible: the normalizer of a Borel subgroup is itself.*

Definition 12. Let G be an algebraic group. Its *radical* $R(G)$ is the unique maximal connected normal solvable subgroup. Its *unipotent radical* $R_{\mathrm{u}}(G)$ is the unique maximal connected normal unipotent subgroup.

The group G is *semisimple* if $R(G)$ is trivial, and *reductive* if $R_{\mathrm{u}}(G)$ is trivial.