18.014–ESG Exam 4

Pramod N. Achar

Fall 1999

1. Determine whether the following sequences and series converge. Where applicable, indicate the type of convergence as well (*i.e.*, conditional, absolute, pointwise, uniform). For problems marked with the symbol "$\triangleright$," compute the limit, if it converges. (*N.B.*: The presence of "$\triangleright$" does *not* imply that the sequence or series in question converges!) Be careful not to confuse sequences with series—only the problems containing a "$\sum$" are series.

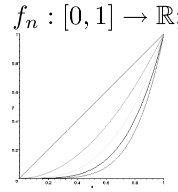
   $\triangleright$ (a) $a_n = \dfrac{n}{n+1} - \dfrac{1}{n}$

   $\triangleright$ (b) $a_n = 1 + (-1)^n$

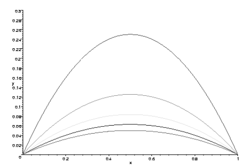
   (c) $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{3^n}$

   (d) $\displaystyle\sum_{n=0}^{\infty} (-1)^n e^{-n^2}$

▷ (e) $f_n : [0,1] \to \mathbb{R}; \quad f_n(x) = x^n$



▷ (f) $f_n : [0,1] \to \mathbb{R}; \quad f_n(x) = \dfrac{x(1-x)}{n}$



2. State the Weierstraß M-test.

3. Consider the following three series of functions:

$$F(x) = \sum_{n=1}^{\infty} c_n \sin nx \qquad G(x) = \sum_{n=1}^{\infty} nc_n \cos nx \qquad H(x) = \sum_{n=1}^{\infty} -\frac{c_n}{n} \cos nx$$

Here, the c_n's are positive constants, with the property that the series $\sum_{n=1}^{\infty} c_n$ converges.

(a) Show that the series for F converges uniformly.

(b) One of G and H will also necessarily converge uniformly by a theorem we proved in class. Which one? Why? (You do not need to give a lengthy justification, but you should also not guess.)

(c) What additional condition would guarantee that the other series of functions (*i.e.*, the one of G and H that was not your answer to the previous question) also converges uniformly? (This should be analogous the convergence of $\sum_{n=1}^{\infty} c_n$ guaranteeing the uniform convergence of F.)

4. (Optional) What is the name of the letter "ß"? What is an acceptable alternate spelling of "Weierstraß" in contexts where the character ß cannot be produced?