18.014–ESG Notes 2

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Fall 1999

1 The Trigonometric Functions

Consider the following properties which might be satisfied by a given pair of functions $u, v : \mathbb{R} \to \mathbb{R}$:

$$Du = v \qquad\qquad Dv = -u \tag{1}$$

$$u(0) = 0 \qquad\qquad v(0) = 1 \tag{2}$$

Theorem 1.1. *There exists a pair of functions u, v satisfying (1) and (2).*

Proof. Deferred. We will do this after we develop some theory of power series. □

Theorem 1.2. *If there is a pair of functions satisfying (1) and (2), it is unique.*

Before we prove this, we need to establish the following:

Lemma 1.3. *Suppose that f and g are two functions such that $Df = g$ and $Dg = -f$. Then $f^2 + g^2$ is a constant.*

Proof. Let us compute the derivative of $f^2 + g^2$:

$$\begin{aligned} D(f^2 + g^2) &= 2f\,Df + 2g\,Dg \\ &= 2f \cdot g + 2g \cdot (-f) \\ &= 0. \end{aligned}$$

Recall that if the derivative of a function is identically 0, then the function must be constant. Thus $f^2 + g^2$ is constant. □

Proof of the uniqueness theorem. Suppose that u and v satisfy (1) and (2); furthermore, suppose that p and q are functions that satisfy analogues of those conditions: i.e.,

$$\begin{aligned} Dp &= q \qquad\qquad & Dq &= -p \\ p(0) &= 0 \qquad\qquad & q(0) &= 1 \end{aligned}$$

Our goal is to show that $u = p$ and $q = v$. To this end, introduce the functions

$$f = u - p \qquad \text{and} \qquad g = v - q.$$

Observe that $Df = Du - Dp = v - q = g$. Similarly, $Dg = Dv - Dq = (-u) - (-p) = -f$. That means we can apply Lemma 1.3 to f and g, and conclude that $f^2 + g^2$ is a constant. What constant is it? We can actually compute the value at $x = 0$, since we know that $u(0) = p(0) = 0$ and $v(0) = q(0) = 1$. Therefore, $f(0) = 0$ and $g(0) = 0$, so

$$f(x)^2 + g(x)^2 = 0$$

for all x. But $f(x)^2$ and $g(x)^2$ must be nonnegative, and the only way in which the sum of two nonnegative numbers can be 0 is that the numbers themselves are 0. That is, $f(x)^2 = 0 = g(x)^2$, so in fact $f(x) = 0 = g(x)$. To say that f and g are identically 0 is precisely to say that $u = p$ and $v = q$. □

Definition 1.4. The unique pair of functions u, v that satisfy (1) and (2) are called *sine* and *cosine*, and denoted by

$$u(x) = \sin x \qquad \text{and} \qquad v(x) = \cos x.$$

Proposition 1.5. $\sin^2 x + \cos^2 x = 1$.

Proof. This is a direct application of Lemma 1.3. That lemma tells us immediately that $\sin^2 x + \cos^2 x$ is a constant; we just need to check what constant it is. We know that $\sin 0 = 0$ and $\cos 0 = 1$, so $\sin^2 0 + \cos^2 0 = 1$. It follows that $\sin^2 x + \cos^2 x = 1$. □

Proposition 1.6. *Addition formulæ for the sine and cosine functions are as follows:*

$$\begin{aligned}\sin(a+b) &= \sin a \cos b + \cos a \sin b \\ \cos(a+b) &= \cos a \cos b - \sin a \sin b\end{aligned}$$

Proof. Let us define two new functions as follows:

$$\begin{aligned}f(x) &= \sin(x+b) - \sin x \cos b - \cos x \sin b \\ g(x) &= \cos(x+b) - \cos x \cos b + \sin x \sin b\end{aligned}$$

Our goal is to show that f and g are identically zero. Let us try to apply Lemma 1.3 to this problem. The derivatives of f and g are:

$$\begin{aligned}Df &= (D\sin)(x+b) - (D\sin)(x)\cos b - (D\cos)(x)\sin b \\ &= \cos(x+b) - \cos x \cos b + \sin x \sin b = g \\ Dg &= (D\cos)(x+b) - (D\cos)(x)\cos b + (D\sin)(x)\sin b \\ &= -\sin(x+b) + \sin x \sin b + \cos x \sin b = -f\end{aligned}$$

Therefore, the lemma tells us that $f^2 + g^2$ is a constant. We can evaluate this at $x = 0$ explicitly. Since $\sin 0 = 0$ and $\cos 0 = 1$, we obtain

$$\begin{aligned}f(0) &= \sin b - \sin 0 \cos b - \cos 0 \sin b = 0 \\ g(0) &= \cos b - \cos 0 \cos b + \sin 0 \sin b = 0\end{aligned}$$

Hence $f^2 + g^2$ is identically zero. It follows that f and g are themselves identically zero. □

2 Exponential and Logarithm

Our approach to these functions is very much like our approach to the trigonometric functions. We were compelled to take the abstract algebraic approach of the preceding section (rather than a geometric one) because we will not even define the plane $\mathbb{R}^2$ until next semester. We could take a simpler route to the exponential and logarithm functions, while still being completely rigorous, by defining the logarithm as the indefinite integral of x^{-1}, and the exponential as the inverse of the logarithm. But there are at least two reasons to address it in the manner we will actually use:

- It reinforces the basic ideas in our discussion of the trigonometric functions.
- It is more intuitive, in the sense that I have always thought of the exponential as a more "basic" function than logarithm. This approach lets us define exponential first, and then the logarithm as its inverse, rather than the other way around.

Let us begin with the following two properties that a function $u : \mathbb{R} \to \mathbb{R}$ might have:

$$Du = u \tag{3}$$

$$u(0) = 1 \tag{4}$$

Theorem 2.1. *There exists a function u satisfying (3) and (4).*

Proof. Deferred. We will do this after we develop some theory of power series. $\square$

Theorem 2.2. *If there is a function satisfying (3) and (4), it is unique.*

Before we prove this, we need to establish the following:

Lemma 2.3. *Suppose that u is a function satisfying (3) and (4). Then $u(a+b) = u(a)u(b)$.*

Proof. Define a new function f by

$$f(x) = u(x+a+b)u(-x).$$

The derivative of f is

$$\begin{aligned} Df &= Du(x+a+b)u(-x) + u(x+a+b)Du(-x)\cdot(-1) \\ &= u(x+a+b)u(-x) - u(x+a+b)u(-x) \\ &= 0. \end{aligned}$$

Therefore, f is constant. What constant is it? We can evaluate f at $x = 0$, using the fact that $u(0) = 1$:

$$f(0) = u(0+a+b)u(0) = u(a+b).$$

That is, $f(x) = u(a+b)$ for all x. Let us write down what this means when $x = -b$:

$$f(-b) = u(a)u(b) = u(a+b).$$

The second equality here is precisely what we wanted to prove. $\square$

Corollary 2.4. *If u satisfies (3) and (4), then $u(x)u(-x) = 1$.* $\square$

Proof of the uniqueness theorem. Suppose that u satisfies (3) and (4); furthermore, suppose that p is a function that satisfies analogues of those conditions; i.e.,

$$\begin{aligned} Dp &= p \\ p(0) &= 1 \end{aligned}$$

Our goal is to show that $u = p$. To this end, introduce the function

$$f(x) = p(x)u(-x).$$

The derivative of f is given by

$$\begin{aligned} Df &= Dp(x)u(-x) + p(x)Du(-x)\cdot(-1) \\ &= p(x)u(-x) - p(x)u(-x) \\ &= 0. \end{aligned}$$

Therefore, f is constant. Since $p(0) = u(0) = 1$, we have $f(0) = 1$, so in fact $f(x) = 1$ for all x. In other words, $p(x)u(-x) = 1$, so

$$p(x) = \frac{1}{u(-x)}.$$

But the preceding corollary implies that $u(x) = 1/u(-x)$, so we conclude that $u = p$. $\square$

Definition 2.5. The unique function u satisfying (3) and (4) is called the *exponential* function, and denoted by

$$u(x) = \exp x.$$

The number $\exp 1$ is called e.

Proposition 2.6. *The exponential function is positive and strictly increasing.*

Proof. We know that $\exp 0$ is positive. Corollary 2.4 says that $\exp x \exp(-x) = 1$ for all x: this implies, in particular, that $\exp x \neq 0$ for all x. Moreover, $\exp x$ can never be negative, for if it were, the Intermediate-Value Theorem (together with the fact that $\exp 0$ is positive) would imply that for some c, $\exp c = 0$, but we just observed that exp is never zero. Since exp is never zero or negative, it is always positive.

Finally, $D\exp$ is always positive as well (because it equals exp), and that means that exp is strictly increasing. $\square$

We already have the notion of raising numbers to rational powers, but the exponential function lets us make sense of raising positive numbers to arbitrary real powers. Before we do that, however, we need to make sure that it agrees with our pre-existing notion of rational powers. This verification was not really done in class, but it is worth doing in writing.

Lemma 2.7. *If n is a nonnegative integer, then $\exp na = (\exp a)^n$.*

Proof. We proceed by induction on n, starting at $n = 0$: we have $\exp(0a) = 1$, which is equal to $(\exp a)^0$.

Now assume the statement is known for $n = k$; we need to prove it for $n = k + 1$:

$$\begin{aligned}
\exp((k+1)a) &= \exp(ka + a) \\
&= \exp ka \exp a && \text{by Lemma 2.3} \\
&= (\exp a)^k \exp a && \text{by the inductive assumption} \\
&= (\exp a)^{k+1}.
\end{aligned}$$

Therefore, $\exp na = (\exp a)^n$ for all nonnegative n. $\square$

Lemma 2.8. *If n is any integer, then $\exp na = (\exp a)^n$.*

Proof. We just need to prove it for n negative. If n is negative, $-n$ is positive, so we have

$$\begin{aligned}
\exp na &= \exp(-n)(-a) \\
&= (\exp(-a))^{-n} && \text{by the preceding lemma} \\
&= \left(\frac{1}{\exp a}\right)^{-n} && \text{by Corollary 2.4} \\
&= (\exp a)^n.
\end{aligned}$$

$\square$

Lemma 2.9. *If n is a positive integer, then $\exp \frac{a}{n} = \sqrt[n]{\exp a}$.*

Proof. We have

$$\begin{aligned}
\left(\exp \frac{a}{n}\right)^n &= \exp\left(n \cdot \frac{a}{n}\right) && \text{by Lemma 2.7} \\
&= \exp a.
\end{aligned}$$

That is, $\exp \frac{a}{n}$ is a positive number whose nth power is $\exp a$, so it must be equal to $\sqrt[n]{\exp a}$. $\square$

Proposition 2.10. *If r is a rational number, then $\exp qa = (\exp a)^q$.*

Proof. We just combine Lemmas 2.8 and 2.9. Suppose that $r = p/q$, where we assume q to be positive. Then, applying those two lemmas, we have

$$\exp \frac{p}{q} a = \sqrt[q]{(\exp a)^p} = (\exp a)^{p/q},$$

as desired. $\square$

In particular, if r is rational, this proposition implies that $\exp r = e^r$. This motivates us to introduce the alternate notation

$$e^x = \exp x.$$

This is the first time we have have ever written (possibly) irrational numbers in an exponent. We will define a^x for arbitrary positive numbers a as well, but we need the logarithm first.

Recall that exp is strictly increasing, continuous, and differentiable. That means it has an inverse which is also strictly increasing, continuous and differentiable. What is the domain of this inverse? The image of exp is the set $\mathbb{R}^+$ of positive numbers, and its domain is all of $\mathbb{R}$, so its inverse ought to have domain $\mathbb{R}^+$ and codomain $\mathbb{R}$.

Definition 2.11. The inverse function of exp is called the *logarithm*, and denoted by $\ln : \mathbb{R}^+ \to \mathbb{R}$.

Now, if r is rational and a is positive, we already knew what the notation a^r means. But now, we also have that $a = e^{\ln a}$, so $a^r = (e^{\ln a})^r = e^{r \ln a}$. On the other hand, $e^{x \ln a}$ is defined for all real x, not just for rational numbers. This leads us to define a^x by the following equation:

$$a^x = e^{x \ln a}.$$

Remember that a is required to be positive, but x can be any real number. (Remember also that negative numbers can only be raised to integer powers—the above formula does not work if a is negative because the logarithm is not defined for negative numbers.)

Proposition 2.12. *The logarithm function has the following properties:*

(a) $D \ln(x) = 1/x$

(b) $\ln(ab) = \ln a + \ln b$

(c) $\ln a^c = c \ln a$

Proof. We evaluate $D \ln$ by our formula for the derivative of an inverse function:

$$D \ln(x) = \frac{1}{D \exp \circ \ln x} = \frac{1}{\exp \circ \ln x} = \frac{1}{x}.$$

Let $x = \ln a$ and $y = \ln b$. In other words, $e^x = a$ and $e^x = b$. We have

$$e^x e^y = e^{x+y}.$$

Taking the logarithm of both sides, we get

$$\ln(e^x e^y) = x + y.$$

Substituting back in for x and y, we get

$$\log(ab) = \log a + \log b.$$

Now, recall the definition of exponentiation with bases other than e:

$$a^c = e^{c \ln a}.$$

Taking the logarithm of both sides, we get

$$\ln a^c = c \ln a.$$

This proves the above identities. ∎

Just as we introduced exponentiation with bases other than e, we can introduce logarithms to bases other than e as well. "Logarithm to the base a," denoted $\log_a$, is the inverse function of a^x. Moreover, just as a^x is expressed in terms of exp (as $e^{x \ln a}$), we can express $\log_a$ in terms of ln. Suppose $y = \log_a x$, so that $a^y = x$. That means

$$e^{y \ln a} = x.$$

Taking the (natural) logarithm of both sides, we get

$$y \ln a = \ln x.$$

Therefore, $y = \ln x / \ln a$. But $y = \log_a x$, so we have the formula

$$\log_a x = \frac{\ln x}{\ln a}.$$

Proposition 2.13. *$D(a^x) = a^x \ln a$ and $D(\log_a x) = 1/(x \ln a)$.*

Proof. These are easy to compute:

$$\begin{aligned} D(a^x) &= D(e^{x \ln a}) \\ &= e^{x \ln a} \cdot \ln a \\ &= a^x \ln a \end{aligned}$$

and

$$\begin{aligned} D(\log_a x) &= D\left(\frac{\ln x}{\ln a}\right) \\ &= \frac{1}{x \ln a}. \end{aligned}$$

Voilà! □

Finally, a note of caution. Often, calculus students are susceptible to making the erroneous calculation $D(a^x) = xa^{x-1}$. **THIS IS WRONG!!!** Anyone making this mistake in this class will be subject to public humiliation!