ESG 18.014 Problem Set 1

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1. Prove Theorem I.7 in Apostol. (You may use Axioms 1–6 and Theorems I.1–I.6.)

2. Prove Theorem I.8 in Apostol. (You may use Axioms 1–6 and Theorems I.1–I.7.)

3. Exercise 4 in Section I 3.3 of Apostol. (You may use Axioms 1–6, Theorems I.1–I.15, and Exercises 2 and 3 in that section.)

4. Show how to define the operations $+$ and $\bullet$ on the three-element set $\mathbb{Z}_3 = \{0, 1, a\}$ so that $\mathbb{Z}_3$ becomes a field.

5. ($*$) Let $\mathbb{Z}_4$ be a set containing four elements: $\{a, b, c, d\}$. Define $+ : \mathbb{Z}_4 \times \mathbb{Z}_4 \to \mathbb{Z}_4$ by the following table:

$$\begin{array}{cccc} a+a=a & a+b=b & a+c=c & a+d=d \\ & b+b=c & b+c=d & b+d=a \\ & & c+c=a & c+d=b \\ & & & d+d=c \end{array}$$

   Assume that $+$ is commutative (that is why, for example, $c+b$ is not given in the table). Show that there is no way to define $\bullet : \mathbb{Z}_4 \times \mathbb{Z}_4 \to \mathbb{Z}_4$ such that $\mathbb{Z}_4$ becomes a field. (*Hint*: Which element is the 0? If there is an operation $\bullet$ satisfying the field axioms, then one of the nonzero elements must be the 1. Show that for each possibility for 1, you can derive a contradiction to one of the theorems I.1–I.15.)