18.014–ESG Problem Set 11

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Monday

1. Let $a : \mathbb{Z}^+ \to \mathbb{R}$ be a sequence, and form the new sequence $\sigma : \mathbb{Z}^+ \to \mathbb{R}$, called its *Cesaro sequence*, or its *sequence of arithmetic means*, by

$$\sigma_n = \frac{a_1 + \cdots + a_n}{n} = \frac{1}{n}\sum_{k=1}^{n} a_k.$$

(a) Show that if $\{a_n\}$ converges, say $\lim_{n\to\infty} a_n = A$, then $\lim_{n\to\infty} \sigma_n = A$ as well. (*Hint*: First prove it in the special case that $A = 0$. You need to show, for any given $\epsilon > 0$, how to choose an $N > 0$ such that if $n \geq N$, then $|\sigma_n| < \epsilon$. Since $\lim_{n\to\infty} a_n = 0$, there is an $M > 0$ such that $n \geq M$ implies $|a_n| < \epsilon/2$. Once you have obtained this M, you can consider the quantity

$$c_n = \frac{a_1 + \cdots + a_{M-1}}{n}.$$

Since the numerator is fixed, the above quantity tends to 0 as $n \to \infty$. In other words, there is a $P > 0$ such that $n \geq P$ implies $|c_n| < \epsilon/2$. Show that if you take N to be the larger of M or P, then it has the desried property; namely, that $n \geq N$ implies $|\sigma_n| < \epsilon$. Finally, prove the statement in the general case, without making any assumptions about A.)

(b) Find an example of sequence $\{a_n\}$ that does not converge, but such that its Cesaro sequence does converge. Compute the limit of its Cesaro sequence.

(c) (Optional) Define a new sequence $b : \mathbb{Z}^+ \to \mathbb{R}$ by $b_n = (n+1)(a_{n+1} - a_n)$. Show that if $\{\sigma_n\}$ converges, and if we further assume that $\{b_n\}$ converges and $\lim_{n\to\infty} b_n = 0$, then $\{a_n\}$ must converge also. (*Hint*: First show that $a_n - \sigma_n = \tau_{n-1}$, where $\{\tau_n\}$ is the Cesaro sequence of $\{b_n\}$. Why must $\{\tau_n\}$ converge?) (In fact, convergence of $\{\sigma_n\}$ implies convergence of $\{a_n\}$ under even weaker hypotheses: $\{b_n\}$ need only be bounded, not necessarily convergent. See Exercise 14 in Chapter 3 of Rudin's *Principles of Mathematical Analysis* for more information.)

2. Prove the following:

   (a) $\lim_{n\to\infty} \left(n \log\left(1 + \frac{r}{n}\right)\right) = r.$

   (b) $\lim_{n\to\infty} \left(1 + \frac{r}{n}\right)^n = e^r.$

3. Exercises 2 in Section 10.4 of Apostol.

4. For what values of a does $\displaystyle\sum_{k=1}^{\infty} \frac{a^n n!}{n^n}$ converge?

Wednesday

5. Exercises 1 and 2 in Section 10.9 of Apostol.

6. Exercises 1 and 4 in Section 10.16 of Apostol.

Friday

7. Exercises 1, 6, and 8 in Section 10.20 of Apostol.

Wednesday

8. Exercises 1, 3, and 9 in Section 11.13 of Apostol. You may find it helpful to refer to the Taylor polynomial formulæ given in Sections 7.4 and 7.8. (These formulæ are given as exercises, but you may use them without proof.)

9. Exercise 24 in Section 11.13 of Apostol. This problem shows that a function may have a Taylor series with an infinite radius of convergence, which nevertheless does not converge to the original function.

10. Let $f : \mathbb{Z}^+ \times \mathbb{R} \to \mathbb{R}$ be the sequence of functions defined by

$$f_n(x) = \frac{x}{1 + nx^2}.$$

    Show that $\{f_n\}$ converges uniformly to a function F. Show also that the sequence of derivatives $\{f_n'\}$ also converges to some function G, but that $F' \neq G$. Thus, even uniform convergence does not guarantee that differentiation is well-behaved with respect to limits of sequences of functions. (*Hint*: Find a sequence of constants M_n such that $|f_n(x)| \leq M_n$ for all n and x, and such that $\lim_{n\to\infty} M_n = 0$.)

Optional

11. Enjoy the winter break! (*Hint*: Don't tool.)

12. What mathematical object is defined by the following? (This is an object that you have heard of, that we will use throughout next semester, but that we have not mentioned at all this semester.)

    > "A function from the Cartesian product of two initial segments of the positive integers to some field, such as $\mathbb{R}$."