18.014–ESG Problem Set 2

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Monday

1. Exercise 2 in Section I 3.5 of Apostol. (You may use Axioms 1–9 and Theorems I.1–I.25.)

2. Let $\mathbb{C}$ denote the field of complex numbers. (Of course, we haven't defined $\mathbb{C}$ rigorously in class; this problem requires you to use what you know about complex numbers in "real life.") Show that there is no subset $\mathbb{C}^+ \subset \mathbb{C}$ with respect to which the order axioms are satisfied. (*Hint*: Suppose there is such a $\mathbb{C}^+$, and consider $i = \sqrt{-1}$. If $i \in \mathbb{C}^+$, where must $-i = i \cdot i \cdot i$ lie? What if $i \notin \mathbb{C}^+$? Derive a contradiction to one of the order axioms.)

3. (Optional) We have assumed in this class that there exists a certain set $\mathbb{R}$, with operations $+$, $\cdot$, that satisfies Axioms 1–10. Given this, define $\mathbb{C}$ rigorously. (You need to describe what the elements of $\mathbb{C}$ are, and then you need to define the operations $+$ and $\cdot$ on it.)

Wednesday

4. Prove the following statements:

   (a) If $x > 0$, then there is a positive integer n such that $1/n < x$.

   (b) If x and y are two real numbers with $x < y$ and $y > 0$, show that there exists a rational number r such that $x < r < y$. (*Hint*: $y - x > 0$, so by the previous part, there is an integer n such that $1/n < y - x$. Define
   $$A = \{q \mid q = m/n \text{ where } m \in \mathbb{Z}^+, \text{ and } q < y\}.$$
   Of course, all the elements of A are rational. Show that A is nonempty and bounded above. Therefore A has a least upper bound b. Show that in fact $b \in A$; i.e., b is the maximum of A. As a consequence, b must be rational. Finally, show that $x < b < y$.)

   (c) If x and y are any two real numbers with $x < y$, show that there exists a rational number r such that $x < r < y$. (This should be easy after the previous part.)

5. Prove that there is no number $x \in \mathbb{Q}$ such that $x^2 = 2$. (See Exercise 11 in Section I 3.12 of Apostol. It is the same problem, and it has a hint. You may use the results in Exercise 10 of that section even though you have not proved them.)

6. Show that $\mathbb{Q}$ does not satisfy the completeness axiom. (*Hint*: Read through the proof that every nonnegative real number a has a nonnegative square root. It uses the set

$$S = \{x \mid x > 0 \text{ and } x^2 < a\}.$$

Now, let $a = 2$, and form the set $T = S \cap \mathbb{Q}$. That is,

$$T = \{x \mid x \in \mathbb{Q},\ x > 0, \text{ and } x^2 < 2\}.$$

T is a subset of $\mathbb{Q}$. Show that T has an upper bound in $\mathbb{Q}$, but that T does not have a least upper bound in $\mathbb{Q}$ (you will need to use Problem 5 to do this). Thus, T is an example of a subset of $\mathbb{Q}$ which violates the completeness axiom.)

Friday

7. Prove that $\sum_{k=1}^{n}(2k-1) = n^2$ by induction.

8. Read Exercise 1 in Section I 4.9 of Apostol. Choose any three of the statements and prove them.

Further comments on Problem 6

You might observe that as a subset of $\mathbb{R}$, the supremum of T is the same as that of S; namely, $\sqrt{2}$, which is not in $\mathbb{Q}$. But this observation is *not* enough to prove that T has no supremum within $\mathbb{Q}$. As an example, consider the following sets:

$$\begin{aligned} A &= \{x \mid 0 \le x < 1\} = [0, 1) \\ B &= \{x \mid 1 \le x < 2\} = [1, 2) \end{aligned}$$

Of course $\sup A = 1$. But if we consider A not as a subset of $\mathbb{R}$, but of the smaller set $\mathbb{R} \setminus B$, then A still has a least upper bound: it is 2.

Similarly, even though $\sup T = \sqrt{2}$, and $\sqrt{2}$ is missing when you examine T in the context of $\mathbb{Q}$ rather than $\mathbb{R}$, you have to account for the possibility that T has a *different* supremum in $\mathbb{Q}$ than in $\mathbb{R}$. That is, you have to prove that that possibility does not occur.