18.014–ESG Problem Set 4

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Monday

1. Prove the linearity property for integrals of bounded functions, as follows:

   (a) Let $A \subset \mathbb{R}$ be nonempty and bounded above, and let $c \in \mathbb{R}$ be some fixed number. Show that if we define $C = \{cx \mid x \in A\}$, then C is nonempty and bounded above, and $\sup C = c \sup A$.

   (b) Let A and B be two nonempty, bounded above subsets of $\mathbb{R}$. Show that if we define $C = \{x + y \mid x \in A \text{ and } y \in B\}$, then C is nonempty and bounded above, and $\sup C = \sup A + \sup B$.

   (c) Using the previous two parts and the linearity property for integrals of step functions, show that if f_1 and f_2 are bounded functions on $[a, b]$ and $c_1, c_2 \in \mathbb{R}$, then

   $$\underline{\int_a^b} (c_1 f_1 + c_2 f_2) = c_1 \underline{\int_a^b} f_1 + c_2 \underline{\int_a^b} f_2.$$

   (This includes showing that $c_1 f_1 + c_2 f_2$ is bounded on $[a, b]$.)

   (d) Show that if f_1 and f_2 are integrable bounded functions on $[a, b]$, then so is $c_1 f_1 + c_2 f_2$, and

   $$\int_a^b (c_1 f_1 + c_2 f_2) = c_1 \int_a^b f_1 + c_2 \int_a^b f_2.$$

   (You will need to use part (c) to do this. You may also use, without proof, the corresponding statement for upper integrals.)

Wednesday

2. (Optional) Prove that between every pair of real numbers, there is an irrational number. (*Hint*: This problem is made somewhat difficult by the fact that the only irrational number we explicitly have a name for so far is $\sqrt{2}$. The goal in this hint will be to add, subtract, and multiply $\sqrt{2}$ with rational numbers until we get something between x and y. Let $x, y \in \mathbb{R}$ with $x < y$. Two weeks ago, you proved that there is a rational number between every two real numbers. Apply this result twice to get

rational numbers q and r such that $x < q < r < y$. Now, define a function $g : [1,2] \to \mathbb{R}$ by $g(t) = (r-q)t + 2q - r$. Show that $g(1) = q$, $g(2) = r$, and that $t_1 < t_2$ implies $g(t_1) < g(t_2)$. Conclude that $q < g(\sqrt{2}) < r$. Finally, show that $g(\sqrt{2})$ is irrational. Then $g(\sqrt{2})$ is the desired irrational number between x and y.)

3. Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

Show that f is discontinuous at every point of $\mathbb{R}$. [You will need to use Problem 2 (even if you did not prove it) and the fact from Problem Set 2 that there is a rational between every two reals.]

4. Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

Show that f is continuous at 0 and discontinuous at every other point of $\mathbb{R}$.

Friday

5. In this problem, you may use the following facts:

$$\lim_{x\to p} \big(f(x) + g(x)\big) = \lim_{x\to p} f(x) + \lim_{x\to p} g(x)$$
$$\lim_{x\to p} \big(f(x)g(x)\big) = \Big(\lim_{x\to p} f(x)\Big)\Big(\lim_{x\to p} g(x)\Big)$$
$$\lim_{x\to p} c = c, \qquad c \text{ a constant.}$$

However, you should not use the definition of limit.

(a) Show that if $\lim_{x\to p} f(x) = A$, then $\lim_{x\to p} -f(x) = -A$.

(b) Show that if $\lim_{x\to p} f(x) = A$ and $\lim_{x\to p} g(x) = B$, then

$$\lim_{x\to p} \big(f(x) - g(x)\big) = A - B.$$

6. Choose three of the limits among the first seven exercises (i.e., in the first column) in Section 3.6 of Apostol, and compute them. Although you need not write out a full proof for each, you should provide some justification by indicating by indicating which theorems you are using.

7. Exercise 21 in Section 3.6 of Apostol.