18.014–ESG Problem Set 5

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Wednesday

1. (Brouwer Fixed-Point Theorem in dimension 1) Let B^1 denote the closed interval $[-1, 1]$. Show that if $f : B^1 \to B^1$ is continuous, then there is some point $x_0 \in B^1$ such that $f(x_0) = x_0$. (Such a point is called a *fixed point* of f.) (*Hint*: Define a function $g : B^1 \to \mathbb{R}$ by $g(x) = f(x) - x$, and apply Bolzano's Theorem to it. Note that the codomain of f is just B^1, not $\mathbb{R}$.)

2. Let P be a polynomial of odd degree. Show that P has at least one real root. (*Hint*: By Bolzano's Theorem, it suffices to find two points $c, d \in \mathbb{R}$ such that $P(c) \leq 0$ and $P(d) \geq 0$. Let us write $P(x) = a_n x^n + \cdots + a_1 x + a_0$, where n is odd and $a_n \neq 0$. For now, assume that $a_n = 1$. Show that if we take
$$d = (n-1)\max\{1, |a_{n-1}|, \ldots, |a_1|, |a_0|\},$$
then $d^n \geq |a_{n-1}d^{n-1} + \cdots + a_1 d + a_0|$. Then show that $P(d) \geq 0$. How can you find a c such that $P(c) \leq 0$? Finally, prove the result in general, using the fact that you know it to be true when $a_n = 1$.)[1]

Friday

3. Exercises 1–5 in Section 3.15 of Apostol.

Notes on Wednesday's Problems

Problem 1 result is a special case of a more general theorem. Let B^n denote an n-dimensional solid ball (something we have of course not defined), and let $f : B^n \to B^n$ be a continuous function. The Brouwer Fixed-Point Theorem says that f must have a fixed point. You just proved it when $n = 1$. If you take 18.901, you will be able to prove it for $n = 2$, and if you take 18.905, you will be able to prove it for all n. A few years ago, I took 18.905 with Prof. Frank Peterson, who is now a septagenarian. He told us the story of how, when he was a graduate student a half century ago, he had attended a talk by Brouwer, who was then an old man. Brouwer in his old age had espoused a philosophy called "intuitionism," which forbids

[1] Thanks to Bobby Kleinberg for suggesting this problem to me.

proof by contradiction. In particular, an intuitionist would accept that $f : B^n \to B^n$ has a fixed point only if you could give a formula for finding the fixed point. So Brouwer gave a talk expounding the falsehood of the Brouwer Fixed-Point Theorem! Prof. Peterson described it to us as "very sad."

Problem 2 is related to the Fundamental Theorem of Algebra, which states that every polynomial with complex coefficients has at least one root in $\mathbb{C}$. Although you have probably heard of this theorem, you have probably not seen a proof of it. One proof of it uses the same techniques from 18.901 that are used to prove the $n = 2$ case of the Brouwer Fixed-Point Theorem. This theorem expresses a property of $\mathbb{C}$ called *algebraic closure*. Of course, $\mathbb{R}$ is not algebraically closed: $x^2 + 1$ is an example of a polynomial with real coefficients but no root in $\mathbb{R}$. Instead, it is only true that *odd*-degree polynomials with real coefficients must have at least one real root.