18.014–ESG Problem Set 6

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Wednesday

A number of theorems that we have proved in class recently (Bolzano, Intermediate-Value, Extreme-Value, Small-Span) all have something to say about a function f under two assumptions: (a) that f is continuous, and (b) that the domain of f is a closed interval. You all know that continuity is an important assumption; the theorems in question fail miserably if you try to apply them to discontinuous functions. The following problems are intended to demonstrate that the assumption about the domain is important too—all the aforementioned theorems can fail for continuous functions if the domain is ill-behaved.

1. Let $A = [a_1, b_1] \cup [a_2, b_2]$ be the union of two disjoint intervals; say $a_1 < b_1 < a_2 < b_2$. For each of the following questions, if the answer is *yes*, give brief justification. If the answer is *no*, give a counterexample.

   (a) If $f : A \to \mathbb{R}$ is continuous, does f necessarily take on every value between $f(a_1)$ and $f(b_2)$?

   (b) If $f : A \to \mathbb{R}$ is continuous, are there points $c, d \in A$ such that $f(c)$ is a maximum of f and $f(d)$ is a minimum?

   (c) If $f : \mathbb{R} \to \mathbb{R}$ is continuous, and x_1 and x_2 are any two points of $\mathbb{R}$, does f necessarily take on every value between $f(x_1)$ and $f(x_2)$?

   (d) If $f : \mathbb{R} \to \mathbb{R}$ is continuous, are there points $c, d \in \mathbb{R}$ such that $f(c)$ is a maximum of f and $f(d)$ is a minimum?

   (*Hint*: Your answers should include two *yes*'s and two *no*'s.) This problem should show you that analogues of both the Intermediate-Value and Extreme-Value Theorems are true under more general assumptions about the domain of f than those under which we proved them. But they are not both true under the *same* more general assumptions.

2. A function $f : B \to \mathbb{R}$ (where $B \subset \mathbb{R}$) is called *uniformly-continuous* if, for any given $\epsilon > 0$, there is a $\delta > 0$ such that if $x_1, x_2 \in B$ and $|x_1 - x_2| < \delta$, then $|f(x_1) - f(x_2)| < \epsilon$.

(a) How is this definition different from that of ordinary continuity? (I know this is a vague question. Your response does not need to be rigorous—I would just like you demonstrate that you understand that continuity and uniform-continuity are not the same.)

(b) Show that if $f : [a,b] \to \mathbb{R}$ is continuous, then it is also uniformly-continuous. (*Hint*: Use the Small-Span Theorem.)

(c) Let $g : (0,1] \to \mathbb{R}$ be given by $g(x) = 1/x$. We know that g is continuous at every point of its domain. Show, however, that g is not uniformly-continuous.

(d) In order for the span of a function on an interval to be defined, the function must have a maximum and a minimum on that interval. Show that for any partition of $(0,1]$, there is a subinterval of the partition on which the span of g is not even defined. Therefore, no analogue of the Small-Span Theorem could be true for g.

Friday

3. Using the following two facts (which we proved in class), derive the quotient rule for differentiation.

$$D(fg) = f\,Dg + g\,Df$$
$$D(1/g) = -Dg/g^2$$

4. Choose three of the derivatives among Exercises 3–12 in Section 4.6 of Apostol, and compute them.