# 18.014–ESG Problem Set 8

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*Monday*

1. Suppose $f : [a, b] \to \mathbb{R}$ is continuous; furthermore, suppose that it is differentiable on $(a, b)$. Show that if $Df(x) > 0$ for all $x \in (a, b)$, then $f$ is strictly increasing. (*Hint*: If $f$ were not strictly increasing, use the Mean-Value Theorem to find a point where $Df$ is zero or negative.)

2. Exercises 19 and 21 from Section 5.5 of Apostol. Both of these exercises involve computing derivatives of functions defined in terms of integrals. But be careful—you cannot apply the First Fundamental Theorem directly to either of them.

*Wednesday*

3. Prove that there exists at least one positive number $a$ such that $\cos a = 0$. (*Hint*: Suppose that $\cos x \neq 0$ for all $x > 0$. Show that $\cos x$ would have to be positive for all $x > 0$. Then show that $\sin x$ would be strictly increasing for positive $x$. It follows that if $x > 0$, then

$$0 = \sin 0 < \sin x < \sin 2x = 2 \sin x \cos x.$$

From this, derive the inequality $(2 \cos x - 1) \sin x > 0$ for $x > 0$. Then, show that $\cos x > 1/2$ for $x > 0$. It follows that

$$\int_0^b \cos x \geq \int_0^b \frac{1}{2}$$

for any $b \geq 0$. Evaluate these integrals, and derive a contradiction.)

4. Let $A = \{x \mid x > 0 \text{ and } \cos x = 0\}$. By the preceding problem, $A$ is nonempty. $A$ is also bounded below (by 0), so we can take its infimum. Let $c = \inf A$. Show that $c \in A$; that is, show that $\cos c = 0$. (*Hint*: Use the sign-preserving property of continuous functions.)

The number $c$ you found in the preceding problem is the smallest positive number whose cosine is 0. We give a special name to twice this number: we define

$$\pi = 2 \inf\{x \mid x > 0 \text{ and } \cos x = 0\}.$$

*Friday*

5. Exercises 3, 8, 11, and 23 from Section 5.8 of Apostol.
6. Exercises 1, 2, and 3 from Section 5.10 of Apostol.