18.03–ESG Notes 1*

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Yesterday in class, the question arose of how one might derive the integrating factor to use in solving a first-order linear equation:

$$\frac{dy}{dx} + P(x)y = Q(x). \tag{1}$$

We want to multiply through by some $\rho(x)$ such that the left-hand side becomes recognizable as the derivative of something. (The right-hand side stays just a function of x, so we can just integrate it.)

$$\rho(x)\frac{dy}{dx} + P(x)\rho(x)y = Q(x)\rho(x) \tag{2}$$

More specifically, we are going to try to obtain the left-hand side as the result of using the product rule for differentiation. Looking at the two terms on the left-hand side, we see that one contains dy/dx and the other contains y, so it is reasonable to suppose that y ought to be one of the factors in the product we're looking for. Moreover, everything else on the left-hand side is just a function of x. So perhaps there is some product of the form $R(x)y$ (where R is an unknown function) such that if we differentiate it with respect to x, we get the left-hand side of (2). In other words, we can rewrite (2) as

$$\frac{d}{dx}(R(x)y) = Q(x)\rho(x) \tag{3}$$

Integrating both sides with respect to x, and then dividing through by $R(x)$, we get

$$y = \frac{1}{R(x)}\left(\int Q(x)\rho(x)\ dx + C\right) \tag{4}$$

But what are $R(x)$ and $\rho(x)$? Let us write out the left-hand side of (3) explicitly:

$$R(x)\frac{dy}{dx} + \frac{dR}{dx}y = Q(x)\rho(x) \tag{5}$$

This is supposed to be just a rewriting of equation (2). Comparing with that equation, we see from the first term that $R(x) = \rho(x)$. The second term tells

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us that $dR/dx = P(x)\rho(x)$; or, using the fact that $R(x) = \rho(x)$,

$$\frac{d\rho}{dx} = P(x)\rho(x). \tag{6}$$

This is a separable differential equation that we can solve for ρ:

$$\int \frac{d\rho}{\rho} = \int P(x)\ dx. \tag{7}$$

We have seen equations of this form many times, and we know the answer right away:

$$\rho(x) = C_1 e^{\int P(x)\ dx}. \tag{8}$$

Now we know the integrating factor $\rho(x)$. As a final step, we can plug back into equation (4) and get the solution

$$y = e^{-\int P(x)\ dx} \left(\int Q(x) e^{\int P(x)\ dx}\ dx + C \right). \tag{9}$$

(Actually, there was a C_1 inside the integral and a $\frac{1}{C_1}$ outside it, but these cancel one another.)