MODULAR CATEGORY $\mathcal{O}$ AND PARITY SHEAVES

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Lecture 1. Let $G = \mathrm{SL}_2$. Identify the weight lattice $\mathbf{X}$ with $\mathbb{Z}$, and the dominant weights $\mathbf{X}^+$ with the nonnegative integers. Under this identification, $\rho = 1$. Assume that $p > 2$.

(1) Which irreducible representations $L(\lambda)$ belong to $\mathscr{A}$? to $\mathscr{B}$?
(2) Find composition series for the objects $\Delta_e, \nabla_e \in \mathcal{O}$.
(3) Show that $L(1) \otimes L(p-1)$ belongs to $\mathscr{A}$, and let $T = \overline{L(1) \otimes L(p-1)}$ be the corresponding object of $\mathcal{O}$. Let s be the nontrivial element of the Weyl group. Show that there are short exact sequences

$$0 \to \Delta_e \to T \to \Delta_s \to 0,$$
$$0 \to \nabla_s \to T \to \nabla_e \to 0.$$

(4) Show that T is both projective and injective. Describe the ring $\mathrm{End}(T)$.
(5) Show that there is an isomorphism of rings

$$\mathrm{End}(L_s \oplus T) \cong \mathrm{End}(\Delta_e \oplus T).$$

(The left-hand side involves the direct sum of all indecomposable tilting objects in $\mathcal{O}$, whereas the right-hand side involves the direct sum of all projective objects. This isomorphism expresses the fact that $\mathcal{O}$ is *self-Ringel-dual.*)
(6) Show that there exactly five indecomposable objects in $\mathcal{O}$, up to isomorphism: namely, L_e, L_s, Δ_e, ∇_e, and T.

Lecture 2. Some of today's exercises use tools from sheaf theory.

(1) Fix an integer $n \geq 2$, and let Y be the space of $n \times n$ nilpotent matrices of rank ≤ 1. Show that $Y \smallsetminus \{0\}$ is a single conjugacy class of matrices. Finally, show that $\dim Y = 2n-2$.
(2) Let

$$\tilde{Y} = \{(x, L) \in Y \times \mathbb{P}^{n-1} \mid L \subset \ker x\}.$$

Show that $\tilde{Y}$ is a vector bundle over $\mathbb{P}^{n-1}$, and hence smooth. Show also that the projection map onto the first factor $\pi : \tilde{Y} \to Y$ is a resolution of singularities.
(3) Compute the stalks of $\pi_* \underline{\Bbbk}_{\tilde{Y}}[2n-2]$. In the case $\Bbbk = \mathbb{C}$, use this to compute the stalks of $\mathrm{IC}(Y)$.
(4) Show that $\tilde{Y}$ is isomorphic as a vector bundle to the cotangent bundle $T^*\mathbb{P}^{n-1}$. (It might be helpful to think of $\mathbb{P}^{n-1}$ as a homogeneous space GL_n/P, where $P \subset \mathrm{GL}_n$ is a suitable parabolic subgroup.) Deduce that its Euler class is $-n$. Finally, explain how $\pi_* \underline{\Bbbk}_{\tilde{Y}}[2n-2]$ decomposes into indecomposable objects depending on the characteristic of $\Bbbk$.

Lecture 3.

(1) Go back and work on some more problems from Lectures 1 and 2!
(2) In Lecture 2, I briefly mentioned Williamson's procedure for producing an integer C and a pair of elements $y, w \in S_N$ such that the stalks $\mathcal{E}_w(\Bbbk)|_{X_y}$ and $\mathrm{IC}_w(\mathbb{C})|_{X_y}$ disagree when p divides C. Go through the combinatorics of [6, §6] and work out some explicit examples. Can you produce an example where C has a prime divisor larger than N? If you are stuck on this, see [5, §6].
(3) Take your example from the previous question, and convert it into a statement about multiplicities in category $\mathcal{O}$. Then lift this to a statement about representations of SL_N. You should name an explicit pair of dominant weights λ, μ for SL_N, along with a list of prime numbers $p > N$, such that $[N(\lambda) : L(\mu)]$ disagrees with the prediction of Lusztig's conjecture in those characteristics.

References

The main reference for Lecture 1 is [4]. For additional background on Weyl modules, dual Weyl modules, translation functors, etc., an excellent source in Jantzen's textbook [2]. The endomorphism ring result from Lecture 3 is based on [1].

The main reference for Lecture 2 is [6]. That paper is a geometric reformulation of the ideas in [5]. For background on the general theory of parity sheaves, see [3].

[1] H. H. Andersen, J. C. Jantzen, and W. Soergel, *Representations of quantum groups at a pth root of unity and of semisimple groups in characteristic p: independence of p*, Astérisque, vol. 220, Soc. Math. France, 1994.

[2] J. C. Jantzen, *Representations of algebraic groups*, 2nd ed., Mathematical Surveys and Monographs, no. 107, Amer. Math. Soc., Providence, RI, 2003.

[3] D. Juteau, C. Mautner, and G. Williamson, *Parity sheaves*, J. Amer. Math. Soc. **27** (2014), 1169–1212.

[4] W. Soergel, *On the relation between intersection cohomology and representation theory in positive characteristic*, Commutative algebra, homological algebra and representation theory (Catania/Genoa/Rome, 1998), J. Pure Appl. Algebra **152** (2000), 311–335.

[5] G. Williamson, *Schubert calculus and torsion explosion*, J. Amer. Math. Soc. **30** (2017), 1023–1046, with a joint appendix with A. Kontorovich and P. J. McNamara.

[6] G. Williamson, *On torsion in the intersection cohomology of Schubert varieties*, J. Algebra **475** (2017), 207–228.

Further Reading

In the lectures, I have described a geometric incarnation of $\mathcal{O}$, which is a kind of "toy model" for $\mathrm{Rep}(G)$. In recent years, there has been quite a lot of progress in developing geometric models for the entire principal block $\mathrm{Rep}_0(G)$: see [8, 9]. The main result of the latter paper is a geometric character formula for tilting modules in $\mathrm{Rep}_0(G)$ (previously conjectured by Riche–Williamson [14]). Parity sheaves play a crucial role in all of this work, as does the theme of *Koszul duality*. For various geometric and representation-theoretic instances of Koszul duality, see [7, 10, 11, 12, 13].

[7] P. Achar and S. Riche, *Modular perverse sheaves on flag varieties II: Koszul duality and formality*, Duke Math. J. **165** (2016), 161–215.

[8] P. Achar and S. Riche, *Reductive groups, the loop Grassmannian, and the Springer resolution*, Invent. Math., to appear, arXiv:1602.04412.

[9] P. Achar, S. Makisumi, S. Riche, and G. Williamson, *Koszul duality for Kac–Moody groups and characters of tilting modules*, J. Amer. Math. Soc., to appear, arXiv:1706.00183.

[10] A. Beĭlinson, V. Ginzburg, and W. Soergel, *Koszul duality patterns in representation theory*, J. Amer. Math. Soc. **9** (1996), 473–527.

[11] R. Bezrukavnikov and Z. Yun, *On Koszul duality for Kac–Moody groups*, Represent. Theory **17** (2013), 1–98.

[12] S. Makisumi, *Modular Koszul duality for Soergel bimodules*, arXiv:1703.01576.

[13] S. Riche, W. Soergel, and G. Williamson, *Modular Koszul duality*, Compos. Math. **150** (2014), 273–332.

[14] S. Riche and G. Williamson, *Tilting modules and the p-canonical basis*, Astérisque **397** (2018).