Exercises on Derived Categories and Perverse Sheaves

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Let $\mathcal{O} = \mathbb{C}[x]$. Regard this as a graded ring by putting $\deg x = 1$. All $\mathcal{O}$-modules below should be assumed to be graded. In particular, $\text{D}^b(\mathcal{O})$ will denote the bounded derived category of graded $\mathcal{O}$-modules.

For any $\mathcal{O}$-modules $M$, we write $M(n)$ for the same module with a shift in grading by $n$. Thus, $\mathcal{O}(n)$ is the free $\mathcal{O}$-module generated by a generator in degree $-n$.

Of course, $\mathcal{O}$-modules are the same as quasicoherent sheaves over $\mathbb{A}^1$. The restriction of an $\mathcal{O}$-module $M$ to the open set $U = \mathbb{A}^1 \smallsetminus \{0\}$ is denoted $M_U$. In particular, we have $\mathcal{O}_U = \mathbb{C}[x, x^{-1}]$.

**Lecture 1: Basics of derived categories**

1. Let $\mathcal{D}$ be a triangulated category that is also abelian, and in which all distinguished triangles are short exact sequences. Prove that $\mathcal{D}$ contains only the zero object.

2. Let $M = \mathcal{O}/(x)$. Check that $R\text{Hom}(M, \mathcal{O}(-1)[1]) \simeq M$.

3. Let $\mathcal{A}$ be an abelian category with enough projectives (or enough injectives). Show that the following conditions are equivalent:
   
   (a) $\text{Ext}^2(M, N) = 0$ for any two objects $M, N \in \mathcal{A}$.
   
   (b) For any $M \in \text{D}^b(\mathcal{A})$, we have $M \simeq \bigoplus H^i(M)[-i]$.

   (Hint: The cone of the zero morphism $C^\bullet \to 0$ is $D^\bullet \oplus C^\bullet[1]$.) A category with these properties is said to be hereditary. Prove that the category of $\mathcal{O}$-modules is hereditary.

4. (Serre–Grothendieck duality for $\mathbb{A}^1$) Prove that the functor $\mathcal{D} = R\text{Hom}(\cdot, \mathcal{O}(-1)[1]) : \text{D}^b(\mathcal{A}) \to \text{D}^b(\mathcal{A})$ is an antiautoequivalence, and that $\mathcal{D}^2 \simeq \text{id} : \text{D}^b(\mathcal{A}) \to \text{D}^b(\mathcal{A})$.

**Lecture 2: $t$-structures and perverse sheaves**

1. Let

   $D^{\leq 0} = \{ M \in \text{D}^b(\mathcal{O}) \mid H^i(M) \text{ is in graded degrees } \geq i, \text{ and } H^i(M_U) = 0 \text{ for } i > 0 \}$,

   $D^{\geq 0} = \{ M \in \text{D}^b(\mathcal{O}) \mid \mathcal{D}(M) \in D^{< 0} \}$.

   Prove that $(D^{\leq 0}, D^{\geq 0})$ is a $t$-structure on $\text{D}^b(\mathcal{O})$. Prove also that every object has finite length, and determine the simple objects.

2. Which indecomposable objects in the heart $\mathcal{C}$ of the $t$-structure above are projective (or injective)? Show that $\mathcal{C}$ is not hereditary, and conclude that $\text{D}^b(\mathcal{C}) \not\simeq \text{D}^b(\mathcal{O})$.

3. Let $X$ denote the 2-sphere, regarded as stratified with a single stratum. Then the category $\mathcal{C}$ of constructible sheaves is just the category of local systems on $X$. Show that $\text{Hom}(\mathcal{S}^2, \mathcal{S}[2]) = 0$ in $\text{D}^b(\mathcal{C})$, but that $\text{Hom}(\mathcal{S}, \mathcal{S}[2]) \simeq \mathcal{S}$ in $\text{D}^b(\mathcal{X})$. Thus, $\text{D}^b(\mathcal{X})$ is not the derived category of $\mathcal{C}$.

**Lecture 3: $\mathcal{D}$-modules and Riemann–Hilbert correspondence**

In this section, let $\mathcal{D} = \mathbb{C}[x, \frac{d}{dx}]$. We regard $\mathcal{D}$ as graded by putting $\deg x = +1$ and $\deg \frac{d}{dx} = -1$. All $\mathcal{D}$-modules are assumed to be graded as well.

1. Determine the indecomposable $\mathcal{D}$-modules. (Up to shift in grading, there are four.) Which of those are simple? (Two of them are simple.) Is the category of $\mathcal{D}$-modules hereditary?

2. For each indecomposable $\mathcal{D}$-module $M$, compute $R\text{Hom}_{\mathcal{D}}(M, \mathcal{O})$ and $R\text{Hom}_{\mathcal{D}}(M_U, \mathcal{O}_U)$.
3. Because we are considering graded modules, it turns out that for all other Zariski open sets $V \subset \mathbb{A}^1$, $R\text{Hom}_{\mathcal{D}_V}(M_V, \mathcal{O}_V)$ is determined by the answers to the previous question. Specifically,

$$R\text{Hom}_{\mathcal{D}_V}(M_V, \mathcal{O}_V) \simeq \begin{cases} 
R\text{Hom}_{\mathcal{D}_U}(M_U, \mathcal{O}_U) & \text{if } V \subset U, \\
R\text{Hom}_{\mathcal{D}}(M, \mathcal{O}) & \text{otherwise.}
\end{cases}$$

Explain why. Then describe the complex of sheaves $R\text{Hom}_{\mathcal{D}}(M, \mathcal{O})$.

4. The four complexes of sheaves obtained in the previous problem are precisely the four indecomposable perverse sheaves on $\mathbb{A}^1$, stratified by $\mathbb{G}_m$-orbits. Identify the simple ones as IC objects. What are the $\text{Ext}^1$-groups among the simple perverse sheaves?

Lecture 4: Weights, purity, and the decomposition theorem

Lecture 5: Applications; perverse coherent sheaves