Exercises on Derived Categories and Perverse Sheaves

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Let $\mathcal{O} = \mathbb{C}[x]$. Regard this as a graded ring by putting $\deg x = 1$. All $\mathcal{O}$-modules below should be assumed to be graded. In particular, $\mathrm{D}^{\mathrm{b}}(\mathcal{O})$ will denote the bounded derived category of graded $\mathcal{O}$-modules.

For any $\mathcal{O}$-modules M, we write $M(n)$ for the same module with a shift in grading by n. Thus, $\mathcal{O}(n)$ is the free $\mathcal{O}$-module generated by a generator in degree $-n$.

Of course, $\mathcal{O}$-modules are the same as quasicoherent sheaves over $\mathbb{A}^1$. The restriction of an $\mathcal{O}$-module M to the open set $U = \mathbb{A}^1 \smallsetminus \{0\}$ is denoted M_U. In particular, we have $\mathcal{O}_U = \mathbb{C}[x, x^{-1}]$.

Lecture 1: Basics of derived categories

1. Let $\mathcal{D}$ be a triangulated category that is also abelian, and in which all distinguished triangles are short exact sequences. Prove that $\mathcal{D}$ contains only the zero object.
2. Let $M = \mathcal{O}/(x)$. Check that $R\mathrm{Hom}(M, \mathcal{O}(-1)[1]) \simeq M$.
3. Let $\mathcal{A}$ be an abelian category with enough projectives (or enough injectives). Show that the following conditions are equivalent:

   (a) $\mathrm{Ext}^2(M, N) = 0$ for any two objects $M, N \in \mathcal{A}$.

   (b) For any $M \in \mathrm{D}^{\mathrm{b}}(\mathcal{A})$, we have $M \simeq \bigoplus H^i(M)[-i]$.

   (*Hint*: The cone of the zero morphism $C^\bullet \xrightarrow{0} D^\bullet$ is $D^\bullet \oplus C^\bullet[1]$.) A category with these properties is said to be *hereditary*. Prove that the category of $\mathcal{O}$-modules is hereditary.
4. (Serre–Grothendieck duality for $\mathbb{A}^1$) Prove that the functor $\mathbb{D} = R\mathrm{Hom}(\cdot, \mathcal{O}(-1)[1]) : \mathrm{D}^{\mathrm{b}}(A) \to \mathrm{D}^{\mathrm{b}}(A)$ is an antiautoequivalence, and that $\mathbb{D}^2 \simeq \mathrm{id} : \mathrm{D}^{\mathrm{b}}(A) \to \mathrm{D}^{\mathrm{b}}(A)$.

Lecture 2: t-structures and perverse sheaves

1. Let

$$D^{\leq 0} = \{M \in \mathrm{D}^{\mathrm{b}}(\mathcal{O}) \mid H^i(M) \text{ is in graded degrees} \geq i, \text{ and } H^i(M_U) = 0 \text{ for } i > 0\},$$
$$D^{\geq 0} = \{M \in \mathrm{D}^{\mathrm{b}}(\mathcal{O}) \mid \mathbb{D}(M) \in D^{\leq 0}\}.$$

   Prove that $(D^{\leq 0}, D^{\geq 0})$ is a t-structure on $\mathrm{D}^{\mathrm{b}}(\mathcal{O})$. Prove also that every object has finite length, and determine the simple objects.
2. Which indecomposable objects in the heart $\mathcal{C}$ of the t-structure above are projective (or injective)? Show that $\mathcal{C}$ is *not* hereditary, and conclude that $\mathrm{D}^{\mathrm{b}}(\mathcal{C}) \not\simeq \mathrm{D}^{\mathrm{b}}(\mathcal{O})$.
3. Let X denote the 2-sphere, regarded as stratified with a single stratum. Then the category $\mathcal{C}$ of constructible sheaves is just the category of local systems on X. Show that $\mathrm{Hom}(\underline{\mathbb{C}}, \underline{\mathbb{C}}[2]) = 0$ in $\mathrm{D}^{\mathrm{b}}(\mathcal{C})$, but that $\mathrm{Hom}(\underline{\mathbb{C}}, \underline{\mathbb{C}}[2]) \simeq \mathbb{C}$ in $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X)$. Thus, $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X)$ is not the derived category of $\mathcal{C}$.

Lecture 3: $\mathcal{D}$-modules and Riemann–Hilbert correspondence

In this section, let $\mathcal{D} = \mathbb{C}[x, \frac{d}{dx}]$. We regard $\mathcal{D}$ as graded by putting $\deg x = +1$ and $\deg \frac{d}{dx} = -1$. All $\mathcal{D}$-modules are assumed to be graded as well.

1. Determine the indecomposable $\mathcal{D}$-modules. (Up to shift in grading, there are four.) Which of those are simple? (Two of them are simple.) Is the category of $\mathcal{D}$-modules hereditary?
2. For each indecomposable $\mathcal{D}$-module M, compute $R\mathrm{Hom}_{\mathcal{D}}(M, \mathcal{O})$ and $R\mathrm{Hom}_{\mathcal{D}_U}(M_U, \mathcal{O}_U)$.

3. Because we are considering *graded* modules, it turns out that for all other Zariski open sets $V \subset \mathbb{A}^1$, $R\mathrm{Hom}_{\mathcal{D}_V}(M_V, \mathcal{O}_V)$ is determined by the answers to the previous question. Specifically,

$$R\mathrm{Hom}_{\mathcal{D}_V}(M_V, \mathcal{O}_V) \simeq \begin{cases} R\mathrm{Hom}_{\mathcal{D}_U}(M_U, \mathcal{O}_U) & \text{if } V \subset U, \\ R\mathrm{Hom}_{\mathcal{D}}(M, \mathcal{O}) & \text{otherwise.} \end{cases}$$

   Explain why. Then describe the complex of sheaves $R\mathcal{H}om_{\mathcal{D}}(M, \mathcal{O})$.

4. The four complexes of sheaves obtained in the previous problem are precisely the four indecomposable perverse sheaves on $\mathbb{A}^1$, stratified by $\mathbb{G}_\mathrm{m}$-orbits. Identify the simple ones as IC objects. What are the Ext^1-groups among the simple perverse sheaves?

Lecture 4: Weights, purity, and the decomposition theorem

Lecture 5: Applications; perverse coherent sheaves