VIR Seminar on Equivariant Cohomology and the Geometric Satake Equivalence

Topics for Junior Participants

- double coset algebras for finite groups
- Hecke algebra for $SL(2)$, perhaps bigger groups
- Construct $SL(2)$-representations on function spaces

Topics for Senior Participants

- Perverse sheaves as a black box
- Example computations
- Convolution product of perverse sheaves on $\mathbb{P}^1$
- Match up results with the undergraduates' work
- Review of the affine Grassmannian
- $\mathbb{C}^\times$-action on affine Grassmannian; fixed-points calculation
- Convolution product of perverse sheaves on the affine Grassmannian for $SL(2)$

Week #1 — 01/25/10

Wed: 2:30-4pm L240 Jr. Thurs. 9:30 - 11:00 am L381(3).

Main Reference:

Junior participants: Aloys Krieg, "Hecke Algebras" Memoirs of the AMS, 1990.

- double coset algebras for finite groups.
- $SL(2,\mathbb{C})$ · figure out Hecke algebra structure.

Aim: produce a write-up of the various topics.

BROADLY

Junior

- Algebras whose elements are functions (on a group or on a set of coset etc).

Senior

- Algebras as Grothendieck rings of categories of sheaves (esp. perverse sheaves)

(Ambes: first lectures on Perverse sheaves).

- understand the tensor category of perverse sheaves on $\mathbb{P}^1$ get a ring: same ring as Junior people.

Later on:

Junior: · reps of $SL(2)$ using spaces of fcns.

Senior: - Review affine Grassmanian (w/out proofs)
- convolution product of perverse sheaves.
- relate to: Junior work.

find relationship: (Geometric Satake Equiv.)

01/25/10.

Goal: Senior: write an exposition on the tensor structure on perverse sheaves on the affine Grassmanian.

Reference: V. Ginzburg "perverse sheaves on a loop group & Langlands duality" 1995
I. Mirkovic', K. Vilonen "Geometric Langlands duality & representations of algebraic groups over comm. rings." Ann. of math 2007

Week#2 — 02|03|10 Amber.

Perverse Sheaves

The Triangle Definition

X - stratified space

$X = \bigsqcup_{i=1}^{k} \Theta_i$ disjoint union of strata.

Θ_i's are manifolds.

Assume Θ_i are simply conn.

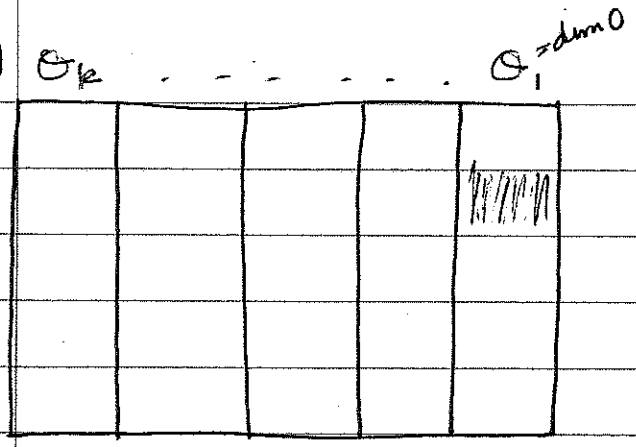
$\dim \Theta_k \geq \dim \Theta_{k-1} \geq \dots \geq \dim \Theta_1$

The category of perverse sheaves $P(X)$ on X is a subcategory of the derived cat. $D(X)$.

Let $\mathcal{F} \in D(X)$

Then we can construct two tables for $\mathcal{F}$.

(T1)



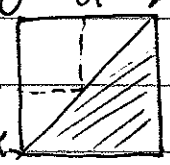
Boxes
col. Θ_j
row i
$H^{-i}(\mathcal{F}|_{\Theta_j})$

simply connected
Boxes have 0 or $\mathbb{C}^m$ for some m. $m \geq 1$.

(T2) Same as T1 except for $\mathbb{D}\mathcal{F}$. $\mathbb{D}\mathbb{D}\mathcal{F} = \mathcal{F}$.

$\mathcal{F}$ is a perverse sheaf if
a) $\dim_{\mathbb{C}} \operatorname{supp} H^{-i}(\mathcal{F}) \leq i$
b) $\dim_{\mathbb{C}} \operatorname{supp} H^{-i}(\mathbb{D}\mathcal{F}) \leq i$

If we have a perverse sheaf table takes form of a triangle.



T1 & T2
$\mathcal{F}$ perverse sh. $\Leftrightarrow$ $\mathbb{D}\mathcal{F}$ is

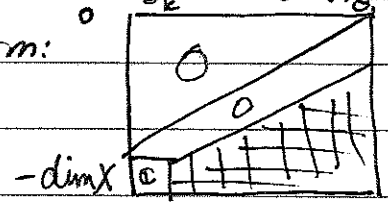
Simple Objects in $P(X)$

Simple means no other perverse sheaves properly contained in the perverse sh or zero.

02/03/10

Simple objects are IC's. $IC(\mathcal{O}_i)$ (simply connected).

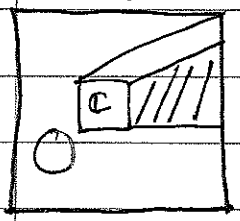
Triangles are of form:



$IC(\mathcal{O}_k)$

- $\dim_{\mathbb{C}} \operatorname{Supp} H^{-j}(IC(\mathcal{O}_i)) \leq j$ $\forall -j \geq -\dim \mathcal{O}_i$

$\leq j$ else



$IC(\mathcal{O}_i)$

- $H^{-j}(IC(\mathcal{O}_i)|_{\mathcal{O}_m}) = 0$

for $\dim \mathcal{O}_m \geq \dim \mathcal{O}_i$ $\mathcal{O}_m \neq \mathcal{O}_i$

- $H^{-j}(IC(\mathcal{O}_i)) = 0$ for $-j < -\dim \mathcal{O}_i$

(and same three conditions for the dual.)

- $H^{-\dim \mathcal{O}_i}(IC(\mathcal{O}_i)|_{\mathcal{O}_i}) = \mathbb{C}$.

The support cond. for $\mathbb{D}f$ gives Δ

IC's are self dual.

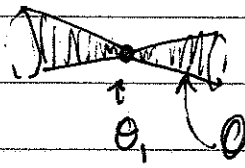
Example: $\mathfrak{sl}_2(\mathbb{C}) = \{2\times 2 \text{ matrices w/ trace } 0\}$,

$\mathcal{N}_{\mathfrak{sl}_2(\mathbb{C})} = \{\text{nilpotent matrices in } \mathfrak{sl}_2(\mathbb{C})\}$.

Strata: the orbits in $\mathcal{N}$ from the $SL_2(\mathbb{C})$.

02/03/10

There are 2 orbits: $\mathcal{O}_1$ & $\mathcal{O}_2$.



"Think $\mathbb{R}^3$" but we're in $\mathbb{C}$.

(not a manifold as a whole but: $\mathcal{O}_1$ manifold
$\mathcal{O}_2$ manifold.)

$\begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathfrak{sl}_2(\mathbb{C})$

for nilpotent want evals $=0$.

$0 = \det \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = -a^2 - bc$ — 2 dim'l.

$a^2 = -bc$. $\dim_{\mathbb{C}} \mathcal{O}_2 = 2$.

in JCF $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. $\dim_{\mathbb{C}} \mathcal{O}_1 = 0$

($a^2 = -bc$ in graph looks like cone.) $\dim \mathcal{N} = 2$.

$\mathcal{O}_2$ $\mathcal{O}_1$

0	0	$\mathbb{C}$
-1	0	0
-2	0	0

$IC(\mathcal{O}_1)$

$P(\mathcal{N})$

	$\mathcal{O}_2$	$\mathcal{O}_1$
0	0	0
-1	0	?
-2	$\mathbb{C}$	?

$IC(\mathcal{O}_2)$

want to determine ?'s.

Properties of maps: $f : Y \to X$ (smooth) X stratified $X = \bigcup \mathcal{O}_i$

Def: f is proper if $f^{-1}(U)$ is compact $\forall$ compact $U \subset X$.

02/03/10

Def: f is *semismall* if $2\dim_{\mathbb{C}} f^{-1}(x) \leq \dim X - \dim \mathcal{O}_x$ where $\mathcal{O}_x$ is strata containing x.

Example: $f: \mathbb{C} \to \mathbb{C}$

" $\{0\} \cup \mathbb{C}\setminus\{0\}$.

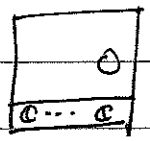
$z \longmapsto z^2$

fibers are finite. is proper.

$\dim f^{-1}(x) \leq 0$ $\quad \dim X - \dim \mathcal{O}_x = \{1, 0\}$

Y, X stratified space. $f: Y \to X$

Let $\underline{\mathbb{C}}_Y$ be the object with triangle



"derived push forward of the constant sheaf"

$f_* \underline{\mathbb{C}}_Y \in D(X)$.

$H^{i-\dim f^{-1}(x)}(f^{-1}(x))$ $\quad x \in \mathcal{O}_x$

$f_* \underline{\mathbb{C}}_Y$ T1 columns given by $H^*(f^{-1}(x))$

"normal cohomology"

with the triangle shifted down

$\dim_{\mathbb{C}}(X)$.

If f is proper then T1 & T2 ↷ like above.

Note: $f_* \underline{\mathbb{C}}_Y$ self-dual.

02/03/10

Decomposition Theorem

Let $f: Y \to X$ be a proper semismall map. Let Y be smooth & $\dim Y = \dim X$.

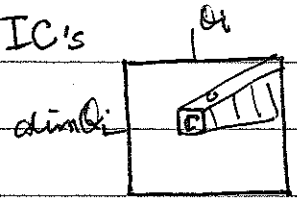
Then, $f_* \underline{\mathbb{C}}_Y$ is a semisimple perv. sh.

Example: $SL_2(\mathbb{C})$

Week #3 — 02/10/10

Amber: Last time:

IC's



$IC(\Theta_i)$.

SL_2

$\mathcal{N}$	Θ_2	Θ_1
0		$\mathbb{C}$
-1		
-2		

$IC(\Theta_1)$

	Θ_2	Θ_1
0	0	0
-1	0	0
-2	$\mathbb{C}$	$\mathbb{C}$

$IC(\Theta_2)$

Assume $f: Y \to X$ is smooth proper & semismall.

Then • $f_* \underline{\mathbb{C}}_Y$ table given by $H^x(f^{-1}(x))$, $x \in \Theta_i$ shifted down $\dim_{\mathbb{C}} X$.

• $f_* \underline{\mathbb{C}}_Y = \oplus$ IC's.

02/10/10

$\tilde{\mathcal{N}} = \{ (x, L) \in \mathcal{N} \times \mathbb{P}^1 \mid xL \subset L \}$

$f \downarrow$ $\mathcal{N}$ $\qquad \downarrow g$ $\mathbb{P}^1$ $\qquad g(x,L) = L$

$f((x,L)) = x.$ (map is proper & semismall.)

$f_* \mathbb{C}_{\tilde{\mathcal{N}}}$

$f^{-1}(x)$

What does a fiber of g look like?

$g^{-1}(L) = \{ x \in \mathcal{N} \mid xL \subset L \}$

e.g. $L = \begin{bmatrix} * \\ 0 \end{bmatrix}$ $\quad g^{-1}(L) = \left\{ \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix} \right\}$

$\theta_1 = 0 \quad f^{-1}(0) \simeq \mathbb{P}^1$

$\theta_2 \ni x \quad f^{-1}(x) = \{pt\}$

$\mathcal{N}$ looks like $\mathbb{P}^1$ w/ line stuck at every pt.

$\dim_{\mathbb{C}} \theta_1 = 0$, $\dim_{\mathbb{C}} \theta_2 = 2 \qquad a^2 + bc = 0.$

Myron: pg 1.

semismall $\quad 2\dim f^{-1}(x) \leq \dim_{\mathbb{C}} X - \dim_{\mathbb{C}} \mathcal{O}$

How do we determine the elts of $\mathbb{P}^1$ preserved by $\mathcal{N}$?

$x \in \mathcal{N} \quad x = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ w/ $a^2 + bc = 0.$ only eval. is 0.

we want $\alpha, \beta \mid \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\qquad \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} a\alpha + b\beta \\ c\alpha - a\beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

we get $(a+c)\alpha + (b-a)\beta = 0.$ $\qquad \alpha, \beta$ not both zero.

If $a = b \Rightarrow a + c = 0$ or $\alpha = 0.$

$a + c = 0 \Rightarrow c = -a \Rightarrow \begin{pmatrix} a & a \\ -a & -a \end{pmatrix} \Rightarrow \beta = -\alpha.$ line $\begin{pmatrix} \alpha \\ -\alpha \end{pmatrix}$

$\alpha = 0, \ \beta \neq 0 \ \& \ a = b \Rightarrow \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}$ kills one dy by $\begin{pmatrix} 0 \\ \beta \end{pmatrix}.$

If $a \neq b$, then the line killed by our matrix is defined by $\begin{pmatrix} a+c & \alpha \\ a-b & \alpha \end{pmatrix}.$

$\Rightarrow$ each $x \in \mathcal{N}$ preserves a unique element of $\mathbb{P}^1$.

02/10/10

Amber

		θ_2	θ_1
$f_*\underline{\mathbb{C}}_{\tilde{\mathcal{N}}}$	0	0	$\mathbb{C}$
	-1	0	0
	-2	$\mathbb{C}$	$\mathbb{C}$

$H^*(pt) \rightsquigarrow H^*(\mathbb{P}^1)$

$H^*(S^2)$

so can fill in $IC(\theta_2)$ table.

proper: $f^{-1}(U)$ cpct $\forall$ U is compact.

$\tilde{\mathcal{N}} \to \mathcal{N}$

$U \subset \mathcal{N}$ cpct

$0 \notin U$: then map is an iso. $f^{-1}(U) \simeq U$.

$0 \in U$:

f is proper.

f is called Springer Resolution.

Myon: pg.2: The orbits of the GL_2-action on $\mathbb{C}^2 \times \mathcal{N}$ are

$\theta_1 = \{(0,0)\}$, $\theta_2 = \{(v,0) \mid v \neq 0\}$, $\theta_3 = \{(0,x) \mid x \neq 0\}$

$\theta_4 = \{(v,x) \mid x, v \neq 0, xv = 0\}$, $\theta_5 = \{(v,x) \mid x \neq 0 \text{ and } xv \neq 0\}$.

$\dim\theta_1 = 0$, $\dim\theta_2 = 2$, $\dim\theta_3 = 2$, $\dim\theta_4 = 3$, $\dim\theta_5 = 4$.

Consider the following resolution: $\mathcal{M}_5 = \{(x, L, v) \mid xL \subset L\}$

$\pi_5 \downarrow$ $\qquad (x,L,v) \mapsto (v,x)$

$\mathbb{C}^2 \times \mathcal{N}$

$\pi_{5*}\underline{\mathbb{C}}_{\mathcal{M}_5}[4]$.

The fiber over a pt in each orbit θ: $\theta_1 \Rightarrow \simeq \mathbb{P}^1$

$\theta_2 \Rightarrow f^{-1}(v,0) \simeq \mathbb{P}^1$ $\qquad \theta_3 \Rightarrow$ fiber is pt $\qquad \theta_4 \Rightarrow$ pt $\qquad \theta_5 \Rightarrow$ pt.

we get table: (B) on next page.

02/10/10

Achar: Topological Fact

ICs have to do w/ topology of closures of strata.

Fact: If $\overline{\mathcal{O}}$ is smooth, then $IC(\overline{\mathcal{O}}) =$ $\underline{0}$ over $\mathbb{C}\,\mathbb{C}\cdots\mathbb{C}$

$\overline{\mathcal{O}_2} = \mathcal{O}_2 \cup \mathcal{O}_1 \simeq \{(v,0)\} \simeq \mathbb{C}^2$ smooth.

(A)

	$\mathcal{O}_5$	$\mathcal{O}_4$	$\mathcal{O}_3$	$\mathcal{O}_2$	$\mathcal{O}_1$
0	0	0	0	0	0
-1	0	0	0	0	0
-2	0	0	0	$\mathbb{C}$	$\mathbb{C}$
-3	0	0	0	0	0
-4	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}$
	$H^*(pt)$			$H^*(\mathbb{P}^1)$	$H^*(\mathbb{P}^2)$

$IC(\mathcal{O}_5)$	$\mathcal{O}_4$	$\mathcal{O}_3$	$\mathcal{O}_2$	$\mathcal{O}_1$
0				0
-1		0	0	0
-2		0	0	0
-3	0	0	0	0
-4	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}$

$IC(\mathcal{O}_2)$	$\mathcal{O}_2$	$\mathcal{O}_1$
0	0	0
-1	0	0
-2	$\mathbb{C}$	$\mathbb{C}$
-3		
-4		

Week #4 | 02 | 17 | 10

Myrm: $\in \mathcal{N} \times \mathbb{P}^1 \times \mathbb{C}^2$

$\mathcal{M}_4 = \{(x, L, v) \mid x \in L \subset L, v \in L\}$

$\downarrow \pi_4$ $\quad (x,L,v) \mapsto (v,x)$

$\mathbb{C}^2 \times \mathcal{N}$

Fibers over pts in orbits.

$\mathcal{O}_1$: $\pi_4^{-1}(0,0) = \{(0,L,0) \mid L \in \mathbb{P}^1\} \simeq \mathbb{P}^1$

$\mathcal{O}_2$: $\pi_4^{-1}(v,0) = \{(0,L,v) \mid v \in L\} = pt$

$\mathcal{O}_3$: $\pi_4^{-1}(0,x) = \{(x,L,0) \mid x \in L \subset L\} = pt$

$\mathcal{O}_4$: $\pi_4^{-1}(v,x) = \{(x,L,v) \mid v \in L\} = pt$

$\mathcal{O}_5$: $\pi_4^{-1}(v,x) = \phi$

	$\mathcal{O}_4$	$\mathcal{O}_3$	$\mathcal{O}_2$	$\mathcal{O}_1$
0	0	0	0	0
-1	0	0	0	$\mathbb{C}$
-2	0	0	0	0
-3	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}$
	$H^*(\mathbb{P}^1)$			

$\dim \mathcal{M}_4 = 3 = \dim \mathcal{O}_4$

$\pi_{4*} \underline{\mathbb{C}}_{\mathcal{M}_4}[3]$

This table only has one IC (only one $\mathbb{C}$ on diag.) it's $IC(\mathcal{O}_4)$.

$IC(\mathcal{O}_4)$

$\pi_{4*}\underline{\mathbb{C}}[3] = IC(\mathcal{O}_4)$. (most interesting IC table so far.)

Achar:

Last time:

Amber $\mathcal{N}$

$f_* \underline{\mathbb{C}}[2] = IC(\mathcal{O}_2) \oplus IC(\mathcal{O}_1)$

Myon $\mathbb{C}^2 \times \mathcal{N}$

$\pi_{5*}\underline{\mathbb{C}}[4]$

$= IC(\mathcal{O}_2) \oplus IC(\mathcal{O}_5)$

(tricky)

02/17/10.

$\in \mathcal{N} \times \mathbb{P}^1$

$\mathcal{M}_3 = \{(X, L) \mid X \subset L \subset L\}$

$\pi_3 \downarrow$ $\quad (X, L)$

$\mathbb{C}^2 \times \mathcal{N} \quad (0, X)$

Fibers:

$\Theta_1 \quad \pi_3^{-1}(0,0) = \{(0, L) \mid L \in \mathbb{P}^1\} \simeq \mathbb{P}^1$

$\Theta_2 \quad \pi_3^{-1}(V, 0) = \{\}$

$\Theta_3 \quad \pi_3^{-1}(V, X) = \{(X, L) \mid X \subset L = 0\} \simeq pt$

$\Theta_4 \quad \pi_3^{-1}(\quad) = \{\}$

$\Theta_5 \quad \pi_3^{-1}(\quad) = \{\}$

	Θ_5	Θ_4	Θ_3	Θ_2	Θ_1
0			0		$\mathbb{C}$
-1			0		0
-2			$\mathbb{C}$		$\mathbb{C}$
-3					
-4					
-5					

$\pi_{3*}\underline{\mathbb{C}}[2]$

$\dim \mathcal{M}_3 = \dim \Theta_3 = 2$

$IC(\Theta_3)$

Θ_3	Θ_1
0	0
0	0
$\mathbb{C}$	$\mathbb{C}$

$IC(\Theta_2)$

Θ_3	Θ_1
0	$\mathbb{C}$
0	0
0	0

Same space as Ambers & same stratification.

02/17/10

Achar: E_8 "mapping of E_8"

Actually

Compute the Kazhdan-Lusztig-Vogan polynomials

Essentially a description of IC tables on flag variety for E_8

w/ 450,000 strata.

Perverse Sheaves on $\mathbb{P}^1$

$G = GL_2(\mathbb{C})$

$B = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \subset GL_2(\mathbb{C})$

Recall: G acts transitively on $\mathbb{P}^1$. (Action: mult

The stabilizer of the line $\begin{bmatrix} * \\ 0 \end{bmatrix}$ is B

so $G/B \simeq \mathbb{P}^1$

Look at B orbits on $\mathbb{P}^1$

(B acts on G/B by multiplying on the left.)

Fact: There are exactly 2 double cosets of B on G.

Explicitly:

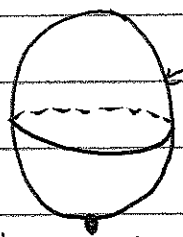
$G = B \cup BsB \qquad s \in \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Equivalent fact:

B has 2 orbits on $\mathbb{P}^1$.

$G/B = \{pt\} \cup$ rest of it.

Picture:



IC's on $\mathbb{P}^1$

$IC(\Theta_0)$

	Θ_1	Θ_0
0	0	$\mathbb{C}$
-1	0	0

$IC(\Theta_1)$

	Θ_1	Θ_0
0	0	0
-1	$\mathbb{C}$	$\mathbb{C}$

b/c $\overline{\Theta_1} = \mathbb{P}^1$ is smooth.

02/17/10

Some other (non-simple) perverse sheaves on $\mathbb{P}^1$

T_1:

	θ_1	θ_2
0	$\mathbb{C}$	$\mathbb{C}$
-1	$\mathbb{C}$	0

Its dual table is:

	θ_1	θ_0
0	0	$\mathbb{C}$
-1	$\mathbb{C}$	$\mathbb{C}$

T_1 is not semisimple.

(Challenge: Write down a composition series for T_1).

Another one:

M:

0	$\mathbb{C}$	dual
$\mathbb{C}$	0	object

0	$\mathbb{C}$
$\mathbb{C}$	0

Write down a composition series for this as well.

GOAL: Describe a way to "multiply" perverse sh. on $\mathbb{P}^1$. (comes from mult in G).

Convolution Diagram

$$G/B \times G/B \xleftarrow{p} G \times G \xrightarrow{\tilde{?}} G \times_B G/B \xrightarrow{m} G/B$$

$\mathbb{P}^1 \times \mathbb{P}^1$

$m(g_1, g_2B) = g_1g_2B$

π

G/B $\quad \pi(g_1, g_2B) = g_1B.$

$G \times_B G/B = G \times^G/B$

$\tilde{?}(g_1g_2B) \sim (g_1, bg_2B)$ for $b \in B$?

or $(g_1, g_2B) \sim (g_1b, b^{-1}g_2B)$

Step ①: prove that

$$(\pi, m) : G \times_B G/B \longrightarrow G/B \times G/B$$

is an isomorphism. (come back to this.)

Step ②: If X & Y are stratified spaces then $X \times Y$ has an obvious strat. consisting of $\mathcal{O}_i^X \times \mathcal{O}_j^Y$

If $\mathcal{F}$ is a perv. sh. on X
$\mathcal{G}$ is a " on Y

can form "external tensor product"

$\mathcal{F} \boxtimes \mathcal{G}$

Table for $\mathcal{F} \boxtimes \mathcal{G}$ is:

In the column labeled by $\mathcal{O}_i^X \times \mathcal{O}_j^Y$ & row k: fill in

$$\bigoplus_{m+n=k} \left(\begin{array}{l}\text{entry in col } \mathcal{O}_i^X \\ \text{row } m \text{ for } \mathcal{F}\end{array}\right) \otimes \left(\begin{array}{l}\text{entry in col } \mathcal{O}_j^Y \\ \text{row } n \text{ for } \mathcal{G}\end{array}\right)$$

(Note: $\mathbb{C}^p \otimes \mathbb{C}^q = \mathbb{C}^{pq}$).
dim'ns mult.)

02/24/10

Convolution Product for Perverse Sheaves on $\mathbb{P}^1$

Let $G = SL_2(\mathbb{C})$. G acts transitively and the space $\mathbb{P}^1$ whose points correspond to lines through the origin in $\mathbb{C}^2$. Moreover, the stabilizer of the line consisting of all vectors of the form $\left[\begin{smallmatrix}*\\0\end{smallmatrix}\right]$ is the group $B = \left\{\left[\begin{smallmatrix}* & *\\ & *\end{smallmatrix}\right]\right\} \subset G$ of upper-triangular matrices. (This is an example of a *Borel subgroup*.) So there is a natural isomorphism $G/B \simeq \mathbb{P}^1$.

1. **Stratification of $\mathbb{P}^1$.** G is the union of two double cosets for B. Specifically, let $s = \left[\begin{smallmatrix}&1\\-1&\end{smallmatrix}\right]$. Then $G = B \cup BsB$. (This is a special case of the *Bruhat decomposition*; the junior people will prove this on the way to studying the Hecke algebra of $SL_2(\mathbb{C})$.) Equivalently, B acts on G/B with two orbits: the single point $1B$ is an orbit, and the rest of $\mathbb{P}^1$ belongs to the B-orbit of the point sB. Let $\mathcal{O}_1 = \{1B\} \subset \mathbb{P}^1$, and let $\mathcal{O}_s$ denote the B-orbit of sB. As a topological space, $\mathcal{O}_s \simeq \mathbb{C}$. Note that $\dim \mathcal{O}_1 = 0$ and $\dim \mathcal{O}_s = 1$.

2. **Perverse sheaves on $\mathbb{P}^1$.** Here are the tables for the simple perverse sheaves $\mathrm{IC}(\mathcal{O}_1)$ and $\mathrm{IC}(\mathcal{O}_s)$.

$\mathrm{IC}(\mathcal{O}_1)$:

	$\mathcal{O}_s$	$\mathcal{O}_1$
0	0	$\mathbb{C}$
-1	0	0

$\mathrm{IC}(\mathcal{O}_s)$:

	$\mathcal{O}_s$	$\mathcal{O}_1$
0	0	0
-1	$\mathbb{C}$	$\mathbb{C}$

The table for $\mathrm{IC}(\mathcal{O}_1)$ follows from the basic properties of perverse sheaves. For $\mathrm{IC}(\mathcal{O}_s)$, we have to use the topological fact that $\overline{\mathcal{O}_s} = \mathbb{P}^1$ is smooth, and that IC's of strata with smooth closures have this easy form. Like all IC's, these perverse sheaves are their own Verdier duals. Here are some other perverse sheaves:

T_s :

	$\mathcal{O}_s$	$\mathcal{O}_1$
0	0	0
-1	$\mathbb{C}$	0

$\mathbb{D}T_s$:

	$\mathcal{O}_s$	$\mathcal{O}_1$
0	0	$\mathbb{C}$
-1	$\mathbb{C}$	$\mathbb{C}$

Y :

	$\mathcal{O}_s$	$\mathcal{O}_1$
0	0	$\mathbb{C}$
-1	$\mathbb{C}$	0

The perverse sheaf T_s is *not* self-dual; its dual $\mathbb{D}T_s$ has a different table. Y is self-dual but is not an IC. Note that $\mathbb{D}T_s$ has the same table as $\mathrm{IC}(\mathcal{O}_1) \oplus \mathrm{IC}(\mathcal{O}_s)$, but they are not isomorphic perverse sheaves.

Technical explanation: let $j : \mathcal{O}_s \to \mathbb{P}^1$ be the inclusion map. Then $T_s = j_!\underline{\mathbb{C}}[1]$ and $\mathbb{D}T_s = j_*\underline{\mathbb{C}}[1]$. Y is an indecomposable *tilting perverse sheaf* in the sense of Beilinson–Bezrukavnikov–Mirković.

3. **External tensor product.** If X and Y are any two stratified spaces, their product $X \times Y$ has an obvious stratification where the strata are just products of strata on X and Y. For example, $\mathbb{P}^1 \times \mathbb{P}^1$ has a stratification with 4 strata (dimensions are given in parentheses):

$$\mathcal{O}_s \times \mathcal{O}_s \ (2), \qquad \mathcal{O}_s \times \mathcal{O}_1 \ (1), \qquad \mathcal{O}_1 \times \mathcal{O}_s \ (1), \qquad \mathcal{O}_1 \times \mathcal{O}_1 \ (0)$$

Suppose $\mathcal{O}_1^X, \ldots, \mathcal{O}_n^X$ are the strata of X, and $\mathcal{O}_1^Y, \ldots, \mathcal{O}_m^Y$ are the strata of Y. Given a perverse sheaf $\mathcal{F}$ on X and a perverse sheaf $\mathcal{G}$ on Y, we can construct a new perverse sheaf on $X \times Y$ denoted $\mathcal{F} \boxtimes \mathcal{G}$, and called their *external tensor product*. Its table of stalks has the property that the column labelled by $\mathcal{O}_i^X \times \mathcal{O}_j^Y$ is the "graded tensor product" of column $\mathcal{O}_i^X$ for $\mathcal{F}$ and column $\mathcal{O}_j^Y$ for $\mathcal{G}$. Recall that for ordinary vector spaces, tensor product just multiplies dimensions: $\mathbb{C}^p \otimes \mathbb{C}^q \simeq \mathbb{C}^{pq}$. For a numbered list (column) of vector spaces, tensor product still multiplies dimensions, but it is also additive on row numbers. That is:

$$\left(\begin{array}{c}\text{entry } r \text{ in column } \mathcal{O}_i^X \times \mathcal{O}_j^Y \\ \text{for } \mathcal{F} \boxtimes \mathcal{G}\end{array}\right) = \bigoplus_{k+l=r} (\text{entry } k \text{ in column } \mathcal{O}_i^X \text{ for } \mathcal{F}) \otimes (\text{entry } l \text{ in column } \mathcal{O}_j^Y \text{ for } \mathcal{G})$$

Here are a couple of tables of external tensor products on $\mathbb{P}^1 \times \mathbb{P}^1$:

$\mathrm{IC}(\mathcal{O}_1) \boxtimes \mathrm{IC}(\mathcal{O}_s)$:

	$\mathcal{O}_s \times \mathcal{O}_s$	$\mathcal{O}_s \times \mathcal{O}_1$	$\mathcal{O}_1 \times \mathcal{O}_s$	$\mathcal{O}_1 \times \mathcal{O}_1$
0	0	0	0	0
-1	0	0	$\mathbb{C}$	$\mathbb{C}$
-2	0	0	0	0

$\mathbb{D}T_s \boxtimes Y$:

	$\mathcal{O}_s \times \mathcal{O}_s$	$\mathcal{O}_s \times \mathcal{O}_1$	$\mathcal{O}_1 \times \mathcal{O}_s$	$\mathcal{O}_1 \times \mathcal{O}_1$
0	0	0	0	$\mathbb{C}$
-1	0	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}$
-2	$\mathbb{C}$	0	$\mathbb{C}$	0

The table for $\mathbb{D}(\mathcal{F} \boxtimes \mathcal{G})$ can be found in similar way, using the fact that $\mathbb{D}(\mathcal{F} \boxtimes \mathcal{G}) \simeq \mathbb{D}\mathcal{F} \boxtimes \mathbb{D}\mathcal{G}$.

Exercise 3.1. Prove that the external tensor product of two perverse sheaves is actually a perverse sheaf. Is is true that $\mathrm{IC}(\mathcal{O}_i^X) \boxtimes \mathrm{IC}(\mathcal{O}_j^Y) \simeq \mathrm{IC}(\mathcal{O}_i^X \times \mathcal{O}_j^Y)$?

Exercise 3.2. Find an example of an external tensor product on $\mathbb{P}^1 \times \mathbb{P}^1$ in which an entry other than 0 or $\mathbb{C}$ occurs somewhere in the table.

4. **Convolution diagram.** Form the following quotient of $G \times G/B$ by an equivalence relation:

$$G \times_B G/B = (G \times G/B)/\sim, \quad \text{where for all } b \in B,\ (g_1 b, g_2 B) \sim (g_1, b g_2 B) \text{ or } (g_1 b, b^{-1} g_2 B) \sim (g_1, g_2 B).$$

Consider the *convolution diagram*

$$\begin{array}{ccccccc} G/B \times G/B & \xleftarrow{p} & G \times G & \xrightarrow{q} & G \times_B G/B & \xrightarrow{m} & G/B \\ & & & & \downarrow \pi & & \\ & & & & G/B & & \end{array}$$

where

$$\begin{aligned} p(g_1, g_2) &= (g_1 B, g_2 B), \\ q(g_1, g_2) &= (g_1, g_2 B), \\ m(g_1, g_2 B) &= g_1 g_2 B, \\ \pi(g_1, g_2 B) &= g_1 B. \end{aligned}$$

The goal is to explain how to take two perverse sheaves $\mathcal{F}$ and $\mathcal{G}$ on G/B and produce a new perverse sheaf $\mathcal{F} \star \mathcal{G}$. This operation on perverse sheaves is called the *convolution product.*

Proposition 4.1. *The map $(\pi, m) : G \times_B G/B \to G/B \times G/B$ is an isomorphism.*

Here is a naive guess on defining the convolution product: the previous proposition lets us think of $G \times_B G/B$ as just an ordinary product, so we can form the external tensor product $\mathcal{F} \boxtimes \mathcal{G}$ on it. Then put $\mathcal{F} \star \mathcal{G} = m_*(\mathcal{F} \boxtimes \mathcal{G})$. **This is wrong:** it doesn't match up with the Hecke algebra in the way it should.

Exercise 4.2. Work out $\mathrm{IC}(\mathcal{O}_s) \star \mathrm{IC}(\mathcal{O}_s)$ using the wrong formula above. To do this, first show that $\mathrm{IC}(\mathcal{O}_s) \boxtimes \mathrm{IC}(\mathcal{O}_s)$ is a "constant sheaf"—its table just has a single row of $\mathbb{C}$'s and 0's everywhere else. Whenever that happens, pushfowards can be computed using the techniques that Amber and Myron demonstrated: determining the fibers of m and writing down their cohomology groups.

5. **Twisted stratification and twisted tensor product.** Consider the following subsets of $G \times_B G/B$:

$$\mathcal{O}_i \,\tilde{\times}\, \mathcal{O}_j = \{(g_1, g_2 B) \mid p(q^{-1}(g_1, g_2 B)) \subset \mathcal{O}_i \times \mathcal{O}_j\}.$$

(Here, $i, j \in \{1, s\}$.) These four sets form a stratification of $G \times_B G/B$, different from the one coming from Proposition 4.1. If $\mathcal{F}$ and $\mathcal{G}$ are perverse sheaves on $\mathbb{P}^1$, there is a perverse sheaf on $G \times_B G/B$ denoted $\mathcal{F} \,\tilde{\boxtimes}\, \mathcal{G}$, called their *twisted external tensor product*, whose table looks just like the one for $\mathcal{F} \boxtimes \mathcal{G}$, but with respect to the new stratification.

Exercise 5.1. Using the isomorphism of Proposition 4.1, write down explicit descriptions of the four strata $\mathcal{O}_s \,\tilde{\times}\, \mathcal{O}_s$, $\mathcal{O}_s \,\tilde{\times}\, \mathcal{O}_1$, $\mathcal{O}_1 \,\tilde{\times}\, \mathcal{O}_s$, $\mathcal{O}_1 \,\tilde{\times}\, \mathcal{O}_1$. In particular, check that the twisted stratification is actually different from the naive one. For any point $x \in G/B$, describe the intersections $m^{-1}(x) \cap (\mathcal{O}_i \,\tilde{\times}\, \mathcal{O}_j)$.

Exercise 5.2. Find an example showing that $\mathcal{F} \,\tilde{\boxtimes}\, \mathcal{G} \neq \mathcal{F} \boxtimes \mathcal{G}$ in general.

Definition 5.3. The convolution product of $\mathcal{F}$ and $\mathcal{G}$ is given by $\mathcal{F} \star \mathcal{G} = m_*(\mathcal{F} \,\tilde{\boxtimes}\, \mathcal{G})$.

Exercise 5.4. Compute as many convolution products as possible. An especially important one is $T_s \star T_s$. Eventually, we will build a multiplication table for T_s and $\mathrm{IC}(\mathcal{O}_1)$. This table will be closely related to the structure of the Hecke algebra for $SL_2(\mathbb{C})$.

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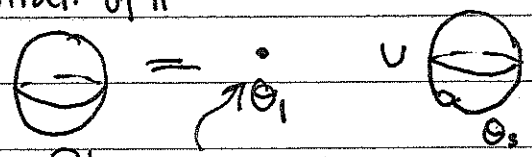
Strat. of $\mathbb{P}^1$



corresp. to the line $[\delta] \subset \mathbb{C}^2$

External tensor product

X, Y are strat. spaces.

X has strata $\theta_1^X, \ldots, \theta_n^X$

Y has strata $\theta_1^Y, \ldots, \theta_m^Y$

Then $X \times Y$ has an obvious stratification w/strata $\theta_i^X \times \theta_j^Y$

If $\mathcal{F}$ is a perverse sheaf on X & $\mathcal{G}$ is " on Y, there is a perverse sh. $\mathcal{F} \boxtimes \mathcal{G}$ on $X \times Y$.

To describe it:

Recall: $\mathbb{C}^p \otimes \mathbb{C}^q = \mathbb{C}^{pq}$ but keep track of row numbers. Row numbers add.

Example: $\mathbb{P}^1$

$\mathbb{D}T_S$:

	θ_s	θ_1
0		$\mathbb{C}$
-1	$\mathbb{C}$	$\mathbb{C}$

Y:

	θ_s	θ_1
0		$\mathbb{C}$
-1	$\mathbb{C}$	

$\mathbb{D}T_S \boxtimes Y$

dim:	2	1	1	0 ← strata of $\mathbb{P}^1 \times \mathbb{P}^1$
	$\theta_s \times \theta_s$	$\theta_s \times \theta_1$	$\theta_1 \times \theta_s$	$\theta_1 \times \theta_1$
0				$\mathbb{C}$
-1		$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}$
-2	$\mathbb{C}$		$\mathbb{C}$	

Recall: An object in the category of sheaves is a perverse sheaf if its table & its dual table both have 0's 'above the diagonal'

1st thing to check: $\mathcal{F} \boxtimes \mathcal{G}$ is actually a perv. sh.?

$\mathcal{F}$ perv. sh. on X and $\mathcal{G}$ perv. sh. on Y.

Look at $\theta_i^X \times \theta_j^Y$.

In table for $\mathcal{F}$ have 0's in column θ_i^X above row $(-\dim \theta_i^X)$

similarly for $\mathcal{G}$ " θ_j^Y .. $(-\dim \theta_j^Y)$

$\dim \theta_i^X \times \theta_j^Y = \dim \theta_i^X + \dim \theta_j^Y$.

so above row $(-\dim \theta_i^X \times \theta_j^Y)$

Can have contribution from row r of $\mathcal{F}$ and row s of $\mathcal{G}$ when $r+s < -\dim \theta_i^X \times \theta_j^Y$.

At least one of these is true.

$r < -\dim \theta_i^X$ or $s < -\dim \theta_j^Y$

so $\left(\begin{matrix}\text{entry } r \\ \text{of } \mathcal{F}\end{matrix}\right) \otimes \left(\begin{matrix}\text{entry } s \\ \text{of } \mathcal{G}\end{matrix}\right) = 0$

at least one of these is 0.

Repeat the argument for $\mathbb{D}(\mathcal{F} \boxtimes \mathcal{G}) \simeq \mathbb{D}\mathcal{F} \boxtimes \mathbb{D}\mathcal{G}$.

so both $\mathcal{F} \boxtimes \mathcal{G}$ & $\mathbb{D}(\mathcal{F} \boxtimes \mathcal{G})$ have 0's above diagonal, so they're perverse sheaves.

Verdier duality: $\mathbb{D}\mathcal{F} = R\mathcal{H}om(\mathcal{F}, \omega)$ ← dualizing complex.

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Recall: IC's have 0's along the diagonal except in one spot. & dual table is the same.

Fact: IC's are actually characterized by these properties.

2nd half of exercise of 3.1 - leave aside from now.

3.2 $\mathbb{D}T_S \boxtimes \mathbb{D}T_S$

	$\Theta_S \times \Theta_S$	$\Theta_S \times \Theta_1$	$\Theta_1 \times \Theta_S$	$\Theta_1 \times \Theta_1$
0				$\mathbb{C}$
-1		$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}^2$
-2	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}$

$G \times_B G/B = (G \times G/B)/\sim$

$(gb, hB) \sim (g, bhB) \quad b \in B$

Recall, $(g_1, g_2 B) \longmapsto g_2 B$ not a well-defined map from $G \times_B G/B \to G/B$

Proposition: $(\pi, m) : G \times_B G/B \to G/B \times G/B$ is an iso.

proof: Let's write down a map in the other direction.

$q : G/B \times G/B \to G \times_B G/B$

$q(g_1 B, g_2 B) = (g_1, g_1^{-1} g_2 B)$

2 things to check: 1) well-defined

2) $(\pi, m) \circ q = id$ & $q \circ (\pi, m) = id$. ?

1) well defined: Choose a different coset rep.

$g_1' = g_1 b$, $g_2' = g_2 c \quad b, c \in B$

Then the formula for q gives

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$(g_1', g_1'g_2'B) = (g_1 b, b^{-1}g_1^{-1}g_2 cB) \overset{b/c\sim}{=} (g_1, g_1^{-1}g_2 cB)$

$\overset{b/c\ c\in B}{=} (g_1, g_1^{-1}g_2 B)$ ✓

$(\pi, m)(g(g_1B, g_2B)) = (g_1B, g_2B)$

$g((\pi,m)(g_1, g_2B)) = g(g_1B, g_1g_2B)$

$= (g_1, g_1^{-1}g_1 g_2 B) = (g_1, g_2B)$. //

□

Give $G \times_B G/B$ a new stratification

$$\Theta_i \tilde{\times} \Theta_j = \{(g_1, g_2B) \mid p(g^{-1}(g_1, g_2B)) \subset \Theta_i \times \Theta_j\}$$

Workout: $\Theta_1 \tilde{\times} \Theta_1$, $\Theta_1 \tilde{\times} \Theta_s$, $\Theta_s \tilde{\times} \Theta_1$, $\Theta_s \tilde{\times} \Theta_s$

Recall, $\Theta_1 \subset \mathbb{P}^1$ is the pt. corresponding to the line $\begin{bmatrix} * \\ 0 \end{bmatrix}$

$\Theta_s \subset \mathbb{P}^1$ is all other lines.

Any line in Θ_s is the span of some vector $\begin{bmatrix} a \\ 1 \end{bmatrix}$ ($a \in \mathbb{C}$) ← pick off this coord.

then get iso. $\Theta_s \xrightarrow{\sim} \mathbb{C}$.

$$\Theta_1 \tilde{\times} \Theta_1 = \{(g_1, g_2B) \mid p(g^{-1}(g_1, g_2B)) = 1B \times 1B \in G/B \times G/B\}$$

$$g^{-1}(g_1, g_2B) = \{(g_1 b_1, b_1^{-1} g_2 c)\} \subset G \times G, \quad b_1 c \in B$$

$$p(g_1 b_1, b_1^{-1} g_2 c) = (g_1 b_1 B, b_1^{-1} g_2 B) = (1B, 1B)$$

$g_1 \in B$, $b_1^{-1} g_2 \in B$ so $g_2 \in B$.

So $(g_1, g_2B) = (1, g_1 g_2 B) = (1, 1B)$

Note: $(\pi, m)(1, 1B) = (1B, 1B) = \Theta_1 \times \Theta_1$

In this case

$$\boxed{\Theta_1 \tilde{\times} \Theta_1 = \Theta_1 \times \Theta_1}$$

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$\Theta_1 \tilde{\times} \Theta_s = \{(g_1, g_2 B) \mid p(q^{-1}(g_1, g_2 B)) \in 1B \times \Theta_s \in G/B \times G/B\}$

$q^{-1}(g_1, g_2 B) = \{(g_1 b, b^{-1} g_2 c)\} \subset G \times G$
$\quad b, c \in G$

$p(q^{-1}(g_1, g_2 B)) = (g_1 B, b^{-1} g_2 B) \in 1B \times \Theta_s$

Then $g_1 \in B$, $b^{-1} g_2 \notin B$ $\quad g_2 \notin B$

$(\pi, m)(g_1, g_2 B) \in \Theta_1 \times \Theta_s$

$\boxed{\Theta_1 \tilde{\times} \Theta_s = \Theta_1 \times \Theta_s}$

work out others for next time:

Week #6 — 03/03/10

Strata on $\mathbb{P}^1 = G/B$

$\Theta_1 = \{1B\} = \{pt.\}$

$\Theta_s = \{gB \mid g \notin B\} \simeq \mathbb{C}$

$G \times_B G/B = \{(g_1, g_2 B)\}/\sim$

Recall, $(\pi, m): G \times_B G/B \xrightarrow{\sim} G/B \times G/B$

so it has a "product strat."

Also has a "twisted strat."

$\Theta_i \tilde{\times} \Theta_j = \{(g_1, g_2 B) \mid p(q^{-1}(g_1, g_2 B)) \subset \Theta_i \times \Theta_j\}$

$$G/B \times G/B \xleftarrow{p} G \times G \xrightarrow{q} G \times_B G/B \xrightarrow{m} G/B$$
$$\downarrow \pi$$
$$G/B$$

Product strata

$\Theta_1 \times \Theta_1 = \{(g_1, g_2 B) \mid g_1 \in B, g_1 g_2 \in B\} = pt$

$\Theta_1 \times \Theta_s = \{(g_1, g_2 B) \mid g_1 \in B, g_1 g_2 \notin B\} \simeq \mathbb{C}$

$\Theta_s \times \Theta_1 = \{(g_1, g_2 B) \mid g_1 \notin B, g_1 g_2 \in B\} \simeq \mathbb{C}$

$\Theta_s \times \Theta_s = \{(g_1, g_2 B) \mid g_1 \notin B, g_1 g_2 \notin B\} \simeq \mathbb{C}^2$

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Twisted strata

dim 0: $\Theta_1 \tilde{\times} \Theta_1 = \{(g_1, g_2B) \mid g_1 \in B, g_2 \in B\} = \Theta_1 \times \Theta_1 = pt$

dim 1: $\Theta_1 \tilde{\times} \Theta_s = \{(g_1, g_2B) \mid g_1 \in B, g_2 \notin B\} = \Theta_1 \times \Theta_s \simeq \mathbb{C}$

dim 1: $\Theta_s \tilde{\times} \Theta_1 = \{(g_1, g_2B) \mid g_1 \notin B, g_2 \in B\} \simeq \mathbb{C}$

$(\Theta_s \tilde{\times} \Theta_1) \cap (\Theta_s \times \Theta_1) = \phi \quad \Theta_s \times \Theta_1 \subset \Theta_s \tilde{\times} \Theta_1 \subset \Theta_s \times \Theta_s$

dim 2: $\Theta_s \tilde{\times} \Theta_s = \{(g_1, g_2B) \mid g_1 \notin B, g_2 \notin B\} \simeq \mathbb{C}^2$

Recall:

$p(g^{-1}(g_1, g_2B))$

$= p(\{(g_1 b, b^{-1} g_2 C)\})$ $\subset G \times G$, $b, c \in B$

$= \{(g_1 bB, b^{-1} g_2 cB) \mid b, c \in B\}$

$= \{(g_1 B, b^{-1} g_2 B) \mid b \in B\}$

$\Theta_s \times \Theta_1 \subset \Theta_s \tilde{\times} \Theta_s$ and $(\Theta_s \tilde{\times} \Theta_s) \cap (\Theta_s \times \Theta_s)$ is nonempty & not all of either stratum.

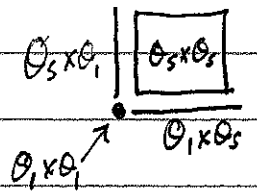
Picture: $G \times_B G/B \simeq G/B \times G/B \simeq \mathbb{P}^1 \times \mathbb{P}^1$

Easier: Replace $\mathbb{P}^1$ by $\mathbb{RP}^1 = \bigcirc$

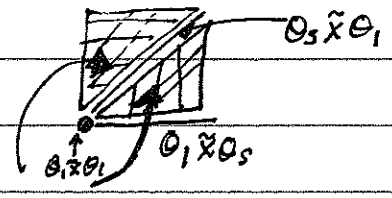
Replace $G/B \times G/B$ by "product of two circles" torus (⊙)

cut open the torus: □

Product Strat:

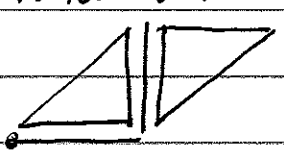


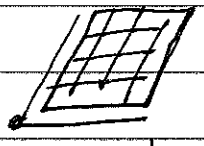
twisted strat.



$\Theta_s \tilde{\times} \Theta_s$

can redraw:





Exercise: Write down twisted strata explicitly in terms of pairs of lines (elts. of $\mathbb{P}^1 \times \mathbb{P}^1$). (Product strata are really products of strata.)

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Is there an automorphism σ of $\mathbb{P}^1 \times \mathbb{P}^1$ that swaps the product strata & the twisted strata?

$$\begin{array}{ccc} \mathbb{P}^1 \times \mathbb{P}^1 & \xrightarrow{m} & \mathbb{P}^1 \\ \sigma? \downarrow & & \downarrow \mathrm{id}? \\ \mathbb{P}^1 \times \mathbb{P}^1 & \xrightarrow[\text{some } \tilde{m}?]{} & \mathbb{P}^1 \end{array}$$

Next step

Write fibers $m^{-1}(x)$ $(x \in G/B)$ in terms of twisted strata.

$m: G \times_B G/B \to G/B$

$(g_1, g_2 B) \mapsto g_1 g_2 B$

For $x \in 1B \in \Theta_1$

$m^{-1}(x) = \Theta_1 \times \Theta_1 \cup \Theta_s \times \Theta_1$

$= \Theta_1 \tilde{\times} \Theta_1 \cup (\text{a line in } \Theta_s \tilde{\times} \Theta_s) \simeq \mathbb{C}$

$m^{-1}(x) \simeq G/B \simeq \mathbb{P}^1$

For $x = gB$ $(x \in \Theta_s)$

$g \notin B$

$\simeq G/B$

$m^{-1}(x) = \Theta_s \times \{x\} \cup \Theta_1 \times \{x\}$ $\quad m^{-1}(x) \simeq \mathbb{P}^1$

$m^{-1}(\Theta_s) = \Theta_1 \times \Theta_s \cup \Theta_s \times \Theta_1$

$= \Theta_1 \tilde{\times} \Theta_s \cup (\Theta_s \tilde{\times} \Theta_1) \cup (\text{a piece of } \Theta_s \tilde{\times} \Theta_s)$

$\mathbb{C}$ $\quad \mathbb{C}$ $\quad = (\Theta_s \tilde{\times} \Theta_s) \cap (\Theta_s \times \Theta_s)$

$\simeq \mathbb{C}^2 \setminus \mathbb{C}$

copy of $\mathbb{C}^2$ w/ line deleted

03/03/10

If $x \in \overset{gB}{\Theta_s}$ $\quad \{(h, h^{-1}gB) \mid h \notin B, h^{-1}g \notin B\}$

$m^{-1}(x) = \underbrace{\{pt\}}_{(1,gB)} \cup \underbrace{\{pt\}}_{(g,1B)} \cup \underbrace{(\mathbb{C} \setminus \{pt\})}$

From: $\Theta_1 \tilde{\times} \Theta_s \quad \Theta_s \tilde{\times} \Theta_1 \quad \Theta_s \tilde{\times} \Theta_s$ $\qquad \mathbb{C} \setminus \{pt\}$ Connected ✝

Let's compute $T_s * T_s = m_*(T_s \tilde{\boxtimes} T_s)$

1st step:

$T_s \tilde{\boxtimes} T_s$

T_s:	Θ_s	Θ_1
0	0	0
-1	$\mathbb{C}$	0

	0
0	0
-1	0
-2	$\mathbb{C}$

	$\Theta_s \tilde{\times} \Theta_s$	$\Theta_s \tilde{\times} \Theta_1$	$\Theta_1 \tilde{\times} \Theta_s$	$\Theta_1 \tilde{\times} \Theta_1$
0	0	0	0	0
-1	0	0	0	0
-2	$\mathbb{C}$	0	0	0

$m_*(T_s \tilde{\boxtimes} T_s)$: get a perv. sh. on G/B

	Θ_s	Θ_1
0		
-1		

Fill in columns with cohomology w/ cpct support of a certain subset of the fibers.

$m_*(T_s \tilde{\boxtimes} T_s)$

	Θ_s	Θ_1
0	$\mathbb{C}$	$\mathbb{C}$
-1	$\mathbb{C}$	0
-2	0	0

$H_c^*(\mathbb{C} \setminus pt)[2]$ $\qquad$ $H_c^*(\mathbb{C})[2]$

use Poincaré duality.

$H^*(\mathbb{C}\setminus pt) = H^*(S^1) = \mathbb{C}$ deg 1, $\mathbb{C}$ deg 0

$H^*_c(\mathbb{C}\setminus\{pt\}) = \mathbb{C}$ deg 2, $\mathbb{C}$ deg 1, 0 deg 0

$H^*_c(\mathbb{R}^n;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \deg n \\ 0 & n-1 \\ \vdots & \\ 0 & 0 \end{cases}$

$\mathbb{C}$ like $\mathbb{R}^2$

"not a perverse sh. b/c $\mathbb{C}$ are not below the diagonal."

	Θ_S	Θ_1
0	$\mathbb{C}$	$\mathbb{C}$
-1	$\mathbb{C}$	0
-2	0	0

Pretend that this table is a direct sum of its entries.

Then: $T_S * T_S \simeq T_S \oplus IC(\Theta_1) \oplus T_S[-1]$

Week #7 — 03/10/10

MV: Strata on $\mathbb{P}^1 = G/B$

$\Theta_1 = \{1B\} = \{pt\}$, $\Theta_S = \{gB \mid g \notin B\} \simeq \mathbb{C}$

Goal: compute $IC(\Theta_1) * T_S$

$T_S * IC(\Theta_1)$

$IC(\Theta_1) * IC(\Theta_1)$

Recall: $IC(\Theta_1)$:

	Θ_S	Θ_1
0	0	$\mathbb{C}$
-1	0	0

T_S:

	Θ_S	Θ_1
0	0	0
-1	$\mathbb{C}$	0

Convolution product: of f and g is given by $f * g = m_*(f \tilde{\boxtimes} g)$.

03/10/10

twisted strata:

dim 0 $\quad \Theta_1 \tilde{\times} \Theta_1 = \Theta_1 \times \Theta_1 = pt$

dim 1 $\quad \Theta_1 \tilde{\times} \Theta_s = \Theta_1 \times \Theta_s \simeq \mathbb{C}$

dim 1 $\quad \Theta_s \tilde{\times} \Theta_1 \simeq \mathbb{C}$

dim 2 $\quad \Theta_s \tilde{\times} \Theta_s \simeq \mathbb{C}^2$

fibers $m^{-1}(x) \qquad x \in G/B$

$x \in \Theta_1 \Rightarrow m^{-1}(x) = \Theta_1 \tilde{\times} \Theta_1 \cup (\underbrace{\text{a line in } \Theta_s \tilde{\times} \Theta_s}_{\simeq \mathbb{C}})$

$x \in \Theta_s \Rightarrow m^{-1}(x) = \underbrace{\{pt\}}_{\Theta_1 \tilde{\times} \Theta_s} \cup \underbrace{\{pt\}}_{\Theta_s \tilde{\times} \Theta_1} \cup \underbrace{(\mathbb{C} \setminus \{pt\})}_{\Theta_s \tilde{\times} \Theta_s}$

Lets compute $IC(\Theta_1) * T_s = m_*(IC(\Theta_1) \tilde{\boxtimes} T_s)$

$IC(\Theta_1) \tilde{\boxtimes} T_s$:

Θ_s	Θ_1	Θ_s	Θ_1	
	$\mathbb{C}$			0
		$\mathbb{C}$		-1
				-2

$\Theta_s \tilde{\times} \Theta_s$	$\Theta_s \tilde{\times} \Theta_1$	$\Theta_1 \tilde{\times} \Theta_s$	$\Theta_1 \tilde{\times} \Theta_1$
		$\mathbb{C}$	

supp $\Theta_1 \tilde{\times} \Theta_s$

$\dim \Theta_1 \tilde{\times} \Theta_s = 1$

so shift of 1

$IC(\Theta_1) * T_s$:

Θ_s	Θ_1
$\mathbb{C}$	

$H_c^*(pt)[1]$ $\qquad$ $H_c^*(\emptyset)[1]$

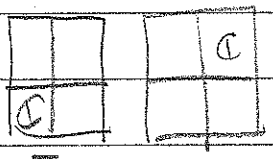
for Θ_s: $m^{-1}(x)$ picks up $\{pt\}$

for Θ_1: $\Theta_1 \tilde{\times} \Theta_s$ doesn't show up in $m^{-1}(x)$ so get $\emptyset$.

03/10/10

$T_s * IC(\mathcal{O}_1) = m_*(T_s \tilde{\boxtimes} IC(\mathcal{O}_1))$.

$T_s \tilde{\boxtimes} IC(\mathcal{O}_1)$:



	$\mathcal{O}_s \tilde{\times} \mathcal{O}_s$	$\mathcal{O}_s \tilde{\times} \mathcal{O}_1$	$\mathcal{O}_1 \tilde{\times} \mathcal{O}_s$	$\mathcal{O}_1 \tilde{\times} \mathcal{O}_1$
0				
-1		$\mathbb{C}$		
-2				

supp $\mathcal{O}_s \tilde{\times} \mathcal{O}_1$

dim $\mathcal{O}_s \tilde{\times} \mathcal{O}_1 = 1$, shift +1.

for $\mathcal{O}_s$: picks up a pt. in $m^{-1}(x)$

for $\mathcal{O}_1$: empty.

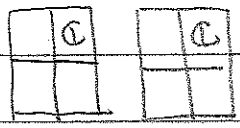
$T_s * IC(\mathcal{O}_1)$:

	$\mathcal{O}_s$	$\mathcal{O}_1$
0		
-1	$\mathbb{C}$	
-2		

$H^*_c(pt)[1] \rightarrow$ $\uparrow H^*_c(\phi)[1]$

$IC(\mathcal{O}_1) * IC(\mathcal{O}_1) = m_*(IC(\mathcal{O}_1) \tilde{\boxtimes} IC(\mathcal{O}_1))$.

$IC(\mathcal{O}_1) \tilde{\boxtimes} IC(\mathcal{O}_1)$:



	$\mathcal{O}_s \tilde{\times} \mathcal{O}_s$	$\mathcal{O}_s \tilde{\times} \mathcal{O}_1$	$\mathcal{O}_1 \tilde{\times} \mathcal{O}_s$	$\mathcal{O}_1 \tilde{\times} \mathcal{O}_1$
0				$\mathbb{C}$
-1				
-2				

$IC(\mathcal{O}_1) * IC(\mathcal{O}_1)$:

	$\mathcal{O}_s$	$\mathcal{O}_1$
0		$\mathbb{C}$
-1		
-2		

$H^*_c(\phi) \nearrow$ $\uparrow H^*_c(pt)$.

supp $\mathcal{O}_1 \tilde{\times} \mathcal{O}_1$

dim 0; no shift

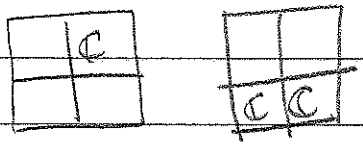
for $\mathcal{O}_s$: empty

for $\mathcal{O}_1$: picks up $\mathcal{O}_1 \tilde{\times} \mathcal{O}_1 \simeq \{pt\}$.

03/10/10

$$IC(\mathcal{O}_1) * IC(\mathcal{O}_s) = m_*(IC(\mathcal{O}_1) \tilde{\boxtimes} IC(\mathcal{O}_s)).$$

$IC(\mathcal{O}_1) \tilde{\boxtimes} IC(\mathcal{O}_s)$:



	$\mathcal{O}_s \tilde{\times} \mathcal{O}_s$	$\mathcal{O}_s \tilde{\times} \mathcal{O}_1$	$\mathcal{O}_1 \tilde{\times} \mathcal{O}_s$	$\mathcal{O}_1 \tilde{\times} \mathcal{O}_1$
0				
-1			$\mathbb{C}$	$\mathbb{C}$
-2				

$IC(\mathcal{O}_1)$ $IC(\mathcal{O}_s)$

$IC(\mathcal{O}_1) * IC(\mathcal{O}_s)$:

	$\mathcal{O}_s$	$\mathcal{O}_1$
0		
-1	$\mathbb{C}$	$\mathbb{C}$
-2		

$H^*_c(pt)[1]$ $H^*_c(pt)[1]$

supp $\mathcal{O}_1 \tilde{\times} \mathcal{O}_s \cup \mathcal{O}_1 \tilde{\times} \mathcal{O}_1$

dim 1 dim 0

dim 1 ; shift = 1.

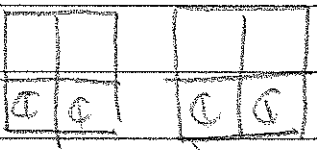
for $\mathcal{O}_s$: $m^{-1}(x)$ picks up $\{pt\}$

for $\mathcal{O}_1$: $m^{-1}(x)$ picks up $\mathcal{O}_1 \tilde{\times} \mathcal{O}_1 \cong pt$

Not presented to crew:

$$IC(\mathcal{O}_s) * IC(\mathcal{O}_s) = m_*(IC(\mathcal{O}_s) \tilde{\boxtimes} IC(\mathcal{O}_s)).$$

$IC(\mathcal{O}_s) \tilde{\boxtimes} IC(\mathcal{O}_s)$:



	$\mathcal{O}_s \tilde{\times} \mathcal{O}_s$	$\mathcal{O}_s \tilde{\times} \mathcal{O}_1$	$\mathcal{O}_1 \tilde{\times} \mathcal{O}_s$	$\mathcal{O}_1 \tilde{\times} \mathcal{O}_1$
0				
-1				
-2	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}$

$IC(\mathcal{O}_s) * IC(\mathcal{O}_s)$:

	$\mathcal{O}_s$	$\mathcal{O}_1$
0	$\mathbb{C}$	$\mathbb{C}$
-1		
-2	$\mathbb{C}$	$\mathbb{C}$

supp $\mathcal{O}_s \tilde{\times} \mathcal{O}_s \cup \mathcal{O}_s \tilde{\times} \mathcal{O}_1 \cup \mathcal{O}_1 \tilde{\times} \mathcal{O}_s \cup \mathcal{O}_1 \tilde{\times} \mathcal{O}_1$

shift : 2 since $\tilde{\boxtimes}$ is constant sheaf.

for $\mathcal{O}_1$: $H^*_c(\mathcal{O}_1 \tilde{\times} \mathcal{O}_1 \cup \mathbb{C} ; \mathbb{C}[2])$

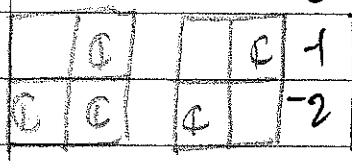
$$H^*_c(\mathbb{P}^1)[2] = H^*(\mathbb{P}^1)[2]$$

$$H^*(\mathbb{P}^1) = \begin{cases} \mathbb{C} & 0 \\ 0 & 1 \\ \mathbb{C} & 2 \end{cases}$$

for $\mathcal{O}_s$: $x \in \mathbb{P}^1$, $x \neq \infty$ fiber $\begin{matrix} x \times \mathbb{P}^1 \\ \downarrow \\ x \end{matrix}$ $H^*_c(\mathbb{P}^1)[2]$

03/10/10

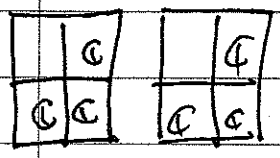
DTs $\tilde{\boxtimes}$ Y :



	$\mathcal{O}_S\tilde{\times}\mathcal{O}_S$	$\mathcal{O}_S\tilde{\times}\mathcal{O}_1$	$\mathcal{O}_1\tilde{\times}\mathcal{O}_S$	$\mathcal{O}_1\tilde{\times}\mathcal{O}_1$
0				
-1		$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}$
-2	$\mathbb{C}$		$\mathbb{C}$	

presented to crew.

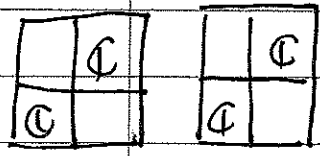
DTs $\tilde{\boxtimes}$ DTs :



	$\mathcal{O}_S\tilde{\times}\mathcal{O}_S$	$\mathcal{O}_S\tilde{\times}\mathcal{O}_1$	$\mathcal{O}_1\tilde{\times}\mathcal{O}_S$	$\mathcal{O}_1\tilde{\times}\mathcal{O}_1$
0				
-1		$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}^2$
-2	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}$

not presented to crew ↓

Y $\tilde{\boxtimes}$ Y :

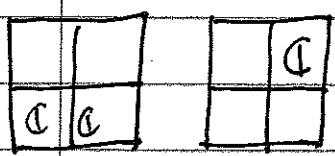


	$\mathcal{O}_S\tilde{\times}\mathcal{O}_S$	$\mathcal{O}_S\tilde{\times}\mathcal{O}_1$	$\mathcal{O}_1\tilde{\times}\mathcal{O}_S$	$\mathcal{O}_1\tilde{\times}\mathcal{O}_1$
0				
-1		$\mathbb{C}$	$\mathbb{C}$	
-2	$\mathbb{C}$			

IC($\mathcal{O}_S$) $\tilde{\boxtimes}$ Y :

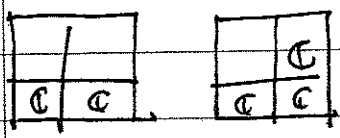
	$\mathcal{O}_S\tilde{\times}\mathcal{O}_S$	$\mathcal{O}_S\tilde{\times}\mathcal{O}_1$	$\mathcal{O}_1\tilde{\times}\mathcal{O}_S$	$\mathcal{O}_1\tilde{\times}\mathcal{O}_1$
0				
-1		$\mathbb{C}$		$\mathbb{C}$
-2	$\mathbb{C}$			

IC($\mathcal{O}_S$) $\tilde{\boxtimes}$ IC($\mathcal{O}_1$) :



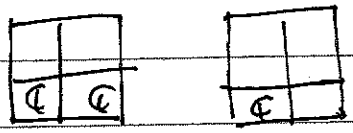
	$\mathcal{O}_S\tilde{\times}\mathcal{O}_S$	$\mathcal{O}_S\tilde{\times}\mathcal{O}_1$	$\mathcal{O}_1\tilde{\times}\mathcal{O}_S$	$\mathcal{O}_1\tilde{\times}\mathcal{O}_1$
0				
-1		$\mathbb{C}$		$\mathbb{C}$
-2				

$IC(\mathcal{O}_S) \tilde{\boxtimes} DT_S$:



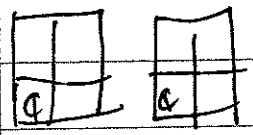
	$\mathcal{O}_S\tilde{\times}\mathcal{O}_S$	$\mathcal{O}_S\tilde{\times}\mathcal{O}_1$	$\mathcal{O}_1\tilde{\times}\mathcal{O}_S$	$\mathcal{O}_1\tilde{\times}\mathcal{O}_1$
0				
-1		$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}$
-2	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}$

$IC(\mathcal{O}_S) \tilde{\boxtimes} T_S$



	$\mathcal{O}_S\tilde{\times}\mathcal{O}_S$	$\mathcal{O}_S\tilde{\times}\mathcal{O}_1$	$\mathcal{O}_1\tilde{\times}\mathcal{O}_S$	$\mathcal{O}_1\tilde{\times}\mathcal{O}_1$
0				
-1				
-2	$\mathbb{C}$		$\mathbb{C}$	

$T_S \tilde{\boxtimes} T_S$



	$\mathcal{O}_S\tilde{\times}\mathcal{O}_S$	$\mathcal{O}_S\tilde{\times}\mathcal{O}_1$	$\mathcal{O}_1\tilde{\times}\mathcal{O}_S$	$\mathcal{O}_1\tilde{\times}\mathcal{O}_1$
0				
-1				
-2	$\mathbb{C}$			

Jeremy: pg. 001 of handout $P_6(\theta)$ ↑ Grothendieck ring of is spherical Hecke algebra $\mathcal{H}(G(k), G(\mathcal{O}))$.

An abelian monoid M is a set with an operation (denoted $+$)

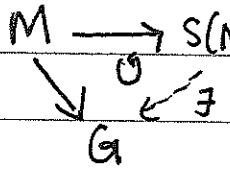
(0) $m+n=n+m$

(1) $\exists 0 \in M$ s.t. $m+0=m$

(2) $l+(m+n) = (l+m)+n$.

03/10/10

The symmetrization $S(M)$ is an abelian group $S(M)$ with a monoid morphism $M \xrightarrow{s} S(M)$ satisfying:



Construction: Consider $M \times M$ with the equivalence relation $(m,n) \sim (m',n')$ iff $\exists\, p \in M$ such that $m+n'+p = m'+n+p$.

(can think: $m-n+p = m'-n'+p$).

Define $S(M) := M \times M/\sim$

Examples: 1) $M = \mathbb{N}$, $S(M) = \mathbb{Z}$

2) $M = \mathbb{Z}\setminus\{0\}$, $S(M) = \mathbb{Q}\setminus\{0\}$.

Proposition:

1) $M \to S(M)$ not. ness. injective.

2) Every element in $S(M)$ is $[m]-[n]$ where $[m] := s(m)$, $[n] := s(n)$.

3) $[m]=[n]$ iff $\exists\, p$. such that $m+p = n+p$.

Let $\mathcal{C}$ = category of f.d. vector spaces over $\mathbb{C}$

Two vector spaces V and W are isom. iff $\dim V = \dim W$.

Let $\mathbb{D}(\mathcal{C})$ denote the set of isom. classes of v.s. in $\mathcal{C}$

$[V]+[W] := [V \oplus W]$.

Then $\mathbb{D}(\mathcal{C})$ is an abelian monoid and $\mathbb{D}(\mathcal{C}) \to \mathbb{N}$

$[V] \longmapsto \dim V$.

is a map of monoids.

03/10/10

Infact an isom. of abelian monoids. (inj & surj.)
so $S(\Phi(\mathcal{C})) = \mathbb{Z}$.

Given a small ab. cat $\mathcal{C}$, then
$S(\Phi(\mathcal{C}))$ is called the Grothendieck group of $\mathcal{C}$.

(from Amber) **Week #8** — 03/17/10

(Mike)

$S \subseteq G$

$Sg = \{sg \mid s \in S\}$

$gS = \{gs \mid s \in S\}$

$SgS = \{s_1 g s_2 \mid s_1, s_2 \in S\}$

$S_1, S_2 \subseteq G$ 'commencerable'

$S_1 \cap S_2$ has finite index in S_1 & S_2 $S_1 \sim S_2$.

$Com(S, G) = \{g \in G \mid g^{-1} S g \sim S\}$. 'commencerator'

$S \subseteq Com(SG) \subseteq G$

(S, G) Heeke pair means $Com(SG) = G$

$R(S, G)$, $R(S, G)^S$, $H(S, G)$

Definition: $R(S, G)$ is the free R-module with all the right cosets Sg rel. to S in G as a basis.

$U = \sum_{g : S \backslash G} u(Sg) Sg$ where $u(Sg) \in R$ & $u(Sg) \neq 0$.

03/17/10

Now, for $r_h \in \mathrm{End}\, R(S,G)$ with $h \in G$

$$r_h(u) := u \cdot h = \sum_{g:S\backslash G} u(S_g) S_g \cdot h$$

Definition: $R(S,G)^S = \{u \in R(S,G) \mid r_s(u) = u \ \forall s \in S\}$.

Definition: $H(S,G)$ like $R(S,G)$ w/ double cosets.

Fact: If $S \lhd G$, $H(S,G) = R(S,G) = R(S,G)^S$.

Double Coset Algebra:

$$R \longrightarrow R^S \longrightarrow H$$
$$\longleftarrow \quad \iota$$

$$\ast : H \times R^S \longrightarrow R^S$$

$$T_i \cdot T_j = \iota^{-1}(T_i \ast \iota(T_j)).$$

Suppose $H(B, GL(n, GF(p^k)))$.

P_π, $\pi \in S_n$

$\tau = \tau_{i,i+1}$

$$BP_\pi B \cdot BP_\tau B = \begin{cases} BP_{\pi\circ\tau}B & \ell(\pi\circ\tau) > \ell(\pi) \\ p^k BP_{\pi\circ\tau}B + (p^k-1)BP_\pi B & \end{cases}$$

$$L(\pi) := \{(i,j) \mid 1 \le i < j \le n, \pi(i) > \pi(j)\} \subseteq S_n \qquad \ell(\pi\circ\tau) < \ell(\pi)$$

$$\ell(\pi) := |L(\pi)|.$$

Theorem: If $K = GF(p^k)$ & $B = B(n,k)$ ← upper matrices, then $H(B, GL(n,k))$ has dim. $n!$ and is gen. by:

B, BP_iB $\quad i = 1, \dots, n-1$.

03/17/10

$T_1 = B\begin{bmatrix}1&0\\0&1\end{bmatrix}B = B$

$T_s = B\begin{bmatrix}0&1\\-1&0\end{bmatrix}B$

	T_1	T_s
T_1	T_1	T_s
T_s	T_s	$p^k T_1 + (p^k - 1)T_s$

(Trevor):

(Humphreys Book)

$R = A = \mathbb{Z}[q, q^{-1}]$

$\mathcal{H}$ be the Hecke Algebra over A as defined before

$i: A \to A$

$q \mapsto q^{-1}$

$i(T_s) := T_{s^{-1}}^{-1} = T_s^{-1}$

Extends linearly to $i: \mathcal{H} \to \mathcal{H}$

$$i^2(T_s) = i(i(T_s)) = i(T_s^{-1}) = i(T_s)^{-1} = (T_s^{-1})^{-1} = T_s.$$

Why does T_s have an inverse?

$T_s \cdot T_s = (q-1)T_s + qT_1$ where $q = |\mathbb{F}_q|$

$T_s \cdot T_s \cdot T_s^{-1} = (q-1)T_s \cdot T_s^{-1} + qT_1 T_s^{-1}$

$T_s = (q-1)T_1 + qT_1 T_s^{-1}$

$T_s - (q-1)T_1 = qT_1 T_s^{-1}$

$q^{-1}(T_s - (q-1)T_1) = T_s^{-1}$

$\Rightarrow T_s^{-1} = q^{-1}T_s - (1 - q^{-1})T_1$

03/17/10

Definition: $R_{x,w} = \sum_{\sigma \in D(x)} (q-1)^{n(\sigma)} q^{m(\sigma)}$

$w = s_1 \cdots s_r \qquad \sigma \in (\sigma_0, \ldots, \sigma_r) \in W^{r+1}$

$\sigma_0 = 1, \quad \sigma_j = \begin{cases} \sigma_{j-1} \\ \sigma_{j-1} s_j \end{cases} \quad (1 \le j \le r)$

$\sigma = s_1, \ldots, \hat{s}_{i_1}, \ldots, \hat{s}_{i_r}, \ldots s_r$

with $\{i_1, \ldots, i_r\} = \{j \mid \sigma_j = \sigma_{j-1}\}$

$n(\sigma) := p$

$m(\sigma) := |\{j \mid \sigma_{j-1} > \sigma_j\}|$

$D(x) = \{\sigma \mid x = \sigma_r \text{ & } \sigma_j \ge \sigma_{j-1} s_j \ \forall\, 1 \le j \le r\}.$

New basis $\{C_w\}$ for $\mathcal{H}$

$\{e_w\}$ will consist of elements fixed under i

write: $T_s^{-1} = q^{-1} T_s - (1 - q^{-1}) T_1$

We also have

$$i(T_s - qT_1) = q^{-1}(T_s - qT_1)$$

We can replace A with a super ring and not change any previous calculations.

Let $A = \mathbb{Z}[q^{1/2}, q^{-1/2}]$

$C_s := q^{-1/2}(T_s - qT_1).$

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Theorem: For each $w \in W$, $\exists!$ element $C_w \in \mathcal{H}$ such that

1) $i(C_w) = C_w$

2) $C_w = \varepsilon_w q^{1/2} \sum_{x \le w} \varepsilon q^{-1} \bar{P}_{x,w} T_x$

where $P_{w,w} = 1$ & $P_{x,w}(q) \in \mathbb{Z}[q]$

$\deg(P_{x,w}(q)) \le \frac{1}{2}(\ell(w) - \ell(x) - 1)$ if $x < w$.

$C_s = (q^{-1/2}) T_s + (-q^{1/2}) T_1$

$C_1 =$ (1) T_1

Kazhdan-Lusztig Poly.

(Pramod)

Theorem: For each stratum $\mathcal{O}_w \in G/B$ $\exists!$ object $I(\mathcal{O}_w) \in D^b_c(G/B)$ such that

1) $\mathbb{D} IC(\mathcal{O}_w) \cong IC(\mathcal{O}_w)$

2) $IC(\mathcal{O}_w)|_{\mathcal{O}_w} = \mathbb{C}[\ell(w)]$ &

$H^{i-\ell(w)}(IC(\mathcal{O}_w)|_{\mathcal{O}_x}) = 0$ if $i < 0$ or if $i > \frac{1}{2}(\ell(w) - \ell(x))$

$\boxed{\ell(w) = \dim \mathcal{O}_w}$

Week #9 — 03/24/10

Jeremy:

The Grothendieck group $K_0(\mathcal{C})$ of an abelian cat. $\mathcal{C}$ is the ~~syn~~ ~~(Grothendieck group)~~ free ~~of the~~ abelian group generated by isomorphism classes of objects in $\mathcal{C}$ modulo the relation

$$[Y] = [X] + [Z] \quad \sim \quad 0 \to X \to Y \to Z \to 0.$$

"behave like split exact seqs."

$\mathrm{Vect}(k) \to \mathbb{N}$

$[V] \mapsto \dim_k V.$

directly from this def. we can see that $K_0(\mathcal{C}) = \mathbb{Z}$.

Sage:

$\mathcal{C} = \mathrm{Vect}(k)$

there is one iso. class for each elt in $\mathbb{N}$.

$X_n := [k^n]$

take the free abelian group & get some relations.

$K_0(\mathcal{C}) =$ free abelian group on X_n's / relations

$\mathbb{Z}^{\mathbb{N}}$

free ab. group on the X_n's

$F = \{$ finite sums $\sum_{n=0}^{\infty} a_n X_n \mid a_i \in \mathbb{Z},\ a_i = 0\ \forall$ bfm i's. $\}$

has ∞ rank want rank 1.

$$F \to K_0(\mathcal{C}) \to$$

$X_2 = 2X_1$ by induction on n.

$X_n = nX_1$ $\quad 0 \to k \to k^2 \to k \to 0 \qquad [k^2] = [k] + [k]$

$b \mapsto (b,0)$ $\qquad \overset{\shortparallel}{X_2} \qquad 2X_1$

$(b,c) \mapsto c$

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$K^{n-1} \to K^n \to K$ $\quad k^{n-1} \to k^n \to k$

$\vec{a} \longrightarrow (\vec{a}, 0)$

$(\vec{a}, b) \mapsto b$

$[K^n] = [K^{n-1}] + [k]$

$= (n-1)[k] + [k] = n[k]$

$n=0.$ $\quad [k^0] = 0$

$0 \to K \to K \to 0 \to 0$

$a \mapsto a$

$X_1 = X_1 + X_0$

$\Rightarrow X_0 = 0.$

$K_0(\mathcal{C})$ is generated by $[X_1]$ as a $\mathbb{Z}$-module.

we've shown: $\quad \mathbb{Z} \to K_0(\mathcal{C})$

$1 \mapsto \overline{X_1}$ ✓

$K_0(\mathcal{C}) \xrightarrow{d} \mathbb{Z}$

$\overline{X_1} \mapsto 1$

$F \xrightarrow{\dim} \mathbb{Z}$ remains to show it kills rels.

$X_i \mapsto i$ $\qquad 0 \to A \to B \to C \to 0$

$B \simeq A \oplus C$ $\qquad [B] - [A] - [C]$

$\dim B = \dim A + \dim C$

$K_0(\mathcal{C}) \simeq \mathbb{Z}.$

"$\mathbb{Z}$ is not just an abelian group but also a ring."

$\mathcal{C} = Vect_{fd}(k)$ 03/24/10

put a ring structure on $K_0(\mathcal{C})$

$V \oplus W$ $\quad \dim V \oplus W = \dim V + \dim W$

$V, W \rightsquigarrow$

$\rightsquigarrow V \otimes W$ $\quad \dim(V \otimes W) = \dim V \dim W$

V bases $e_1, \dots, e_n$

W $\quad f_1 \dots f_m$

$V \otimes W$ a vs. w/ basis $e_i \otimes f_j$

$V \otimes W = \{ \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij}\, e_i \otimes f_j \}$ independent of choice of basis.

bilinear maps $(\varphi : V \times W \to k)^*$ is actually the dual but $V^* \simeq V$ in $\mathcal{C}$.

$\varphi(v, w) \in k$

$\varphi(a_1 v_1 + a_2 v_2, w) = a_1 \varphi(v_1, w) + a_2 \varphi(v_2, w)$

$\dots$ sim. on sec. term

$e_i^* \otimes f_j^* : (e_{i'} \otimes f_{j'}) = \delta_{ii'} \delta_{jj'}$

$V \simeq V'$ and $W \simeq W'$

$V \otimes W \simeq V' \otimes W$

in Groth. ~~$[V \otimes W] = [V]$~~

$[V] \cdot [W] = [V \otimes W]$ well defined product.

for ring struct. need prod to be ass.

$(V \otimes W) \otimes U \simeq V \otimes (W \otimes U)$

In this case: $V \otimes W \simeq W \otimes V$

mult. identity: $k \otimes V \simeq V \simeq V \otimes k$ ← unit in Grothendieck group

$[k][V] = [V] = [V][k]$

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need dist. prop.

$(V_1 \oplus V_2) \otimes W \simeq (V_1 \otimes W) \oplus (V_2 \otimes W)$ for V.S.

$(\dim V_1 + \dim V_2)\dim W$

$\dim V_1 \dim W + \dim V_2 \dim W$

$([V_1] + [V_2])[W] \simeq [V_1][W] + [V_2][W]$

true for V.S. but nontriv for modules

$K_0(\mathcal{C})$ inherits a ring structure from $\otimes$ prod.

$K_0(\mathcal{C}) \simeq \mathcal{H}$ are ring isom.

tensor cat $\mathcal{C} \longrightarrow K_0(\mathcal{C})$ ring struct.

(V, σ) $\sigma: V \to V$ lin

$\sigma^2 = I_V$.

(*)
$$\begin{array}{ccc} V & \xrightarrow{\sigma} & V \\ f \downarrow & & \downarrow f \\ W & \xrightarrow{\tau} & W \end{array}$$

$f \in \mathrm{Hom}((V,\sigma),(W,\tau))$

$\{f \in \mathrm{Hom}(V,W)$ s.t. (*) commutes$\}$.

$(V,\sigma) \oplus (W,\tau) = (V \oplus W, (\sigma \oplus \tau))$

$(V,\sigma) \otimes (W,\tau) = (V \otimes W, \sigma \otimes \tau)$

check is an involution.

$(\sigma \otimes \tau)(e_i \otimes f_j) = \sigma(e_i) \otimes \tau(f_j) = \sum_{i'} \sigma_{i'i} e_{i'} \otimes \sum_{j'} \tau_{j'j} f_{j'} = \sum_{i'j'} \sigma_{i'i}\tau_{j'j}(e_{i'} \otimes f_{j'})$

$\sigma = (\sigma_{ij})$ $\tau = (\tau_{ij})$ $\sigma(e_i) = \sum \sigma_{i'i} e_{i'}$ $\tau(f_j) = \sum \tau_{j'j}(f_{j'})$

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$\mathcal{C}$ = cat of f.d. v.s. w/ invol.

Exercise: Compute the ring $K_0(\mathcal{C})$

1st step classify the isom. classes.

$(k,1)$ and $(k,-1)$ diff 1-dim

If you have an involution will be diag.
look at the eval's.

Achar. $\mathcal{P}(\mathbb{P}^1)$ - category of perverse sheaves on $\mathbb{P}^1$.

{ Groth. gp

$K_0(\mathcal{P}(\mathbb{P}^1))$

The convolution product $*$
behaves like $\otimes$ prod in v.s.
and makes the Groth. gp into a ring.

CAUTION: Even though $\mathcal{F} * \mathcal{G}$ (perv. sh.)
might not be a perverse sheaf,
it still gives you a well-defined element
$[\mathcal{F} * \mathcal{G}] \in K_0(\mathcal{P}(\mathbb{P}^1))$.
(Need: the language of triangulated cats and
derived cats.)

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$K_0(\mathcal{P}(\mathcal{P}'))$ = the Hecke algebra $\mathcal{H}$ from last week.

[Perv. sh. T_s] ⟶ Hecke alg. elt. called T_s

[$IC(\mathcal{O}_1)$] ⟶ Hecke alg. elt. T_1 $= C_1$
(mult. identity in this ring $\mathcal{H}$)

[$IC(\mathcal{O}_s)$] ⟶ Kazhdan Lusztig basis elements C_s.

proof of this: C_w's are characterized by some list of algebraic properties

$IC(\mathcal{O}_w)$'s - - - - - prop's.

These properties match up.

Practical consequence: can turn the problem of describing IC's (fill in table - hard in general) into the problem of describing C_w's in terms of T_w's.
(i.e. finding coeffs for a of change of basis — computational problem that can be done by an algorithm.)
→ K-L poly's.

Chris: Question: is there an isomorphism $\varphi: \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$

such that $\varphi^{-1}(\mathcal{O}_s \times \mathcal{O}_1) = \mathcal{O}_s \tilde{\times} \mathcal{O}_1$

$\varphi^{-1}(\mathcal{O}_1 \times \mathcal{O}_s) = \mathcal{O}_1 \tilde{\times} \mathcal{O}_s$? Ans: No.

$\mathbb{P}^1 = SL_2/B$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to (a:c)$

SL_2

$\mathbb{P}^1 = (x:y)/\sim$ s.t. x,y not simultaneously 0.

& $(x:y) \sim (ax:ay)$ $a \in \mathbb{C}^\times$

$SL_2 \times_B SL_2/B \to \mathbb{P}^1 \times \mathbb{P}^1$

$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot b, b^{-1} \begin{pmatrix} e & f \\ g & h \end{pmatrix} \cdot B \right) \longmapsto \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} B, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} \right)$

g_1 g_2

So, $(g_1 B, g_2 B) \in \mathbb{P}^1 \times \mathbb{P}^1$ is in $\mathcal{O} \tilde{\times} \mathcal{O}'$

whenever $g_1 \in \mathcal{O}$ & $g_1^{-1} \cdot g_2 \in \mathcal{O}'$.

$g_1^{-1} \cdot g_2 = \begin{pmatrix} d & b \\ -c & a \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} * & * \\ g \cdot a - e \cdot c & * \end{pmatrix}$

$P = [(a:c), (e:g)] \in \mathbb{P}^1 \times \mathbb{P}^1$

$\mathcal{O}_1 \tilde{\times} \mathcal{O}_1$: $c = 0$, $ag - ec = 0$

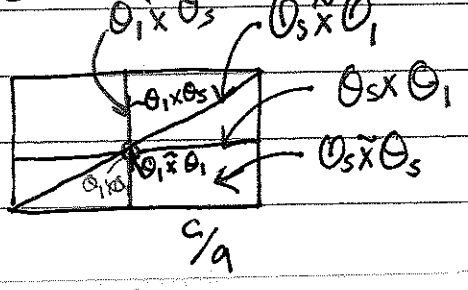
$\star$ $\mathcal{O}_s \tilde{\times} \mathcal{O}_1$: $c \neq 0$, $ag - ec = 0$

$\mathcal{O}_1 \tilde{\times} \mathcal{O}_s$: $c = 0$, $ag - ec \neq 0$

$\mathcal{O}_s \tilde{\times} \mathcal{O}_s$: $c \neq 0$, $ag - ec \neq 0$

$(\mathbb{P}^1 - \{0\} \times \mathbb{P}^1 - \{0\})$ $0:(0:c)$

$\frac{g}{e}$



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If $\varphi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$

then $\varphi^{-1}(\mathbb{P}^1 \times U_1) = U_s \tilde{\times} U_1 \cup U_1 \tilde{\times} U_1$

$\varphi^{-1}(U_1 \times \mathbb{P}^1) = U_1 \tilde{\times} U_s \cup U_1 \tilde{\times} U_1$

$\varphi = \varphi_1 \times \varphi_2$

$\varphi_2, \varphi_1 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$

$\varphi(x,y) = (\varphi_1(x,y), \varphi_2(x,y))$.

φ_1:

$\varphi_1^{-1}(U_1) = \varphi^{-1}(U_1 \times \mathbb{P}^1)$

$\varphi_1^{-1}(x,y) = x$

$x \in U_1 \Rightarrow (x,y) \in U_1 \times \mathbb{P}^1$

$\varphi_1^{-1}(U_1) = U_1 \times \mathbb{P}^1$

φ_2: $\varphi^{-1}(\mathbb{P}^1 \times U_1) = U_s \tilde{\times} U_1 \cup U_1 \times U_1$

$\Rightarrow \varphi_2^{-1}(U_1) =$ "

$\varphi_2((\overset{x}{a:c}), (\overset{y}{e:g})) = (\alpha(a:c),(e:g), \beta(a:c),(e:g))$

α & β should be polynomials in a, c, e, g.

① α & β need to be homogeneous of same degree.

$\deg_x(\alpha) = \deg_x(\beta)$ same for $\deg_y(\alpha) = \deg_y(\beta)$.

'bi homogeneous'

$\varphi_2^{-1}((u:0)) = \{ae - cg = 0\}$

$\beta(x,y) = 0$ whenever $ac - eg = 0$

$\Rightarrow \beta(x,y) = (ag - ce)^m$.

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① $\deg_x(\beta) = 1 = \deg_y(\beta)$.

$\alpha(x,y)$ bihomogeneous polynomial
with $\deg_x(\alpha) = \deg_y(\alpha) = m$.

exercise: show that if α has the form given,
then α & β share a common 0 on $\mathbb{P}^1 \times \mathbb{P}^1$.
- break into two cases.
1) assume $c, g \neq 0$
2) assume $c = g = 0$.

$(\alpha \cdot \beta)$

If $\alpha(x,y) = \beta(x,y) = 0$

1) $\{\alpha(x,y) = 0\} = \varphi_2^{-1}(0)$

$\{\beta(x,y) = 0\} = \varphi_2^{-1}(\infty)$

More general condition: When does there exist a map from X to $\mathbb{P}^1$, where X is proj. alg. var?

Related question: Can we get from $\mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}^1 \times \mathbb{P}^1$ by a sequence of semismall maps such that takes $IC(\mathbb{C} \times \mathbb{C}')$ to $IC(\mathbb{C} \tilde{\times} \mathbb{C}')$.

Definition (Blow up)

$\widetilde{\mathbb{P}^1 \times \mathbb{P}^1} \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$

$[a:c][e:g][y:v]$

subset cut out by $(ag - ec)y - cg \cdot v \neq 0$

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$X=(a:c)$
$y=(e:g)$
$z=(u:v)$

$\mathbb{P}^1\times\mathbb{P}^1 \xrightarrow{\pi_1} \mathbb{P}^1\times\mathbb{P}^1$

$x,y,z \longrightarrow (x,y)$

Fibers: 1) $c\neq 0$ $\quad x\notin \mathcal{O}_1$

$\pi_1^{-1}(x,y) = (x,y,z)$

~~1~~ conditions on z

$(ag-ec)\cdot u - cgv = 0$

$\left(\frac{a}{c}g - e\right)u = g\cdot v$ determines a single point.

① $g=0$, $e\neq 0$, $eu=0 \Rightarrow u=0$. $\Rightarrow (0:v)$

② $g\neq 0$ ... single pt on fiber.

Fibers of π_1 $\quad c\neq 0$ single point

$g\neq 0$ single point.

2] $c=g=0$ $\quad \infty\infty$

$(\mathcal{O}_1\times\mathbb{P}^1)$ $\quad (x,y,z)$ $z\in\mathbb{P}^1$

$(ag-ec)\cdot u - c\cdot g\cdot v = 0$

$\Rightarrow 0=0.$

e.g. can check π_1 is semismall.

$\mathbb{P}^1\times\mathbb{P}^1 \xrightarrow{\pi_2} \mathbb{P}^1\times\mathbb{P}^2$

$(x,y,z) \longmapsto (x,z)$

$\pi_2^{-1}(\mathcal{O}_1\times\mathbb{P}^1) = \{(a:0),(e:g),(u:v)\}$

$(ag-ec)u - cgv = 0$

sidenote:

Problem: $\mathbb{P}^1(\mathbb{R})\times\mathbb{P}^1(\mathbb{R}) \simeq S^1\times S^1$

ex2: use group law to construct φ.

$\mathbb{P}^1(\mathbb{C})\times\mathbb{P}^1(\mathbb{C}) \cong S^2\times S^2$

No group law.

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$u \neq 0 \Rightarrow \pi_2^{-1}(\mathcal{O}_1 \times \mathcal{O}_s) \Rightarrow g = 0$

exercise 3: Calculate $\pi_2^{-1}(\mathcal{O}_1 \times \mathcal{O}_s)$

$\pi_2^{-1}(\mathcal{O}_s \times \mathcal{O}_1) \ldots$

and compare with $\pi_1^{-1}(\mathcal{O}_1 \tilde{\times} \mathcal{O}_s)$ & $\pi_1^{-1}(\mathcal{O}_s \tilde{\times} \mathcal{O}_1)$.

exercise 4: Question

Look up constraints on a topological having 'Homotopy' group structure

show that no flag variety is an H-space.

No flag varieties are H-space.

Affine Grassmanian

Lattices in an F-Vector Space

$\mathcal{O}$ ring of power series with complex coeffs.

$p \in \mathcal{O}$

$$p = a_0 + a_1 t + a_2 t^2 + \cdots + a_j t^j + \cdots$$

Fact: p is invertible iff $a_0 \neq 0$.

Exercise.

$\Rightarrow F$ = 'field' of Laurant series

$$f \in F \quad f = a_{-r} t^{-r} + a_{-r+1} t^{-r+1} + \cdots + a_0 + a_1 t + \cdots$$

$f^{-1} = t^r \cdot (t^r f)^{-1}$ as long as $a_{-r} \neq 0$.

Check that this is true.

V is an n-dimensional F-vector space.

Choose a basis: $e_1, \ldots, e_n$ and

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$V = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$ such that $f_i \in F$

A lattice $L \subset V$ is the $\mathcal{O}$-span of n linearly independent vectors.

e.g. $L^\circ = \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix}$ $p_i \in \mathcal{O}$: $\mathcal{O}$-span of $e_1, \dots, e_n$

e.g. pick $v_1, \dots, v_n \in V$ that are linearly independent,

$L = \text{span}_{\mathcal{O}}(v_1, \dots, v_n)$

$(v_1, \dots v_n) \in End_F(V) \qquad M_n(F)$

Un end means $\det \neq 0$.

$\therefore$ Sets of n linearly independent vectors

$\updownarrow$

elements of $GL_n(F)$.

$GL_n(F)$ acts on the space of lattices

$g \in GL_n(F) \qquad g \cdot \text{span}(v_1, \dots, v_n) = \text{span}(g \cdot v_1, \dots, g \cdot v_n)$.

In fact, $GL_n(F)$ acts transitively on space of lattices.

& stabilizes of standard lattice

(elements of $GL_n(F)$ such that $g \cdot L^\circ = L^\circ$). $\quad \mathcal{O} = GL_n(\mathcal{O})$

$\Rightarrow$ if Gr^{aff} = affine Grassmanian = (topological) space of lattices in V,

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$$Gr^{aff} \cong GL_n(F)/GL_n(\mathcal{O}).$$

Week #11 — 04/14/10

Sage:

Affine Grassmanian

$\mathcal{O} = \mathbb{C}[[t]]$ ring of power series

$F = \mathbb{C}((t))$ field of Laurant series

$val(f)$ = deg at which f starts $\quad val(0) = -\infty$.

$f = a_k t^k + a_{k+1} t^{k+1} + \dots$

$a_k \neq 0.$

$f = t^k (a_k + a_{k+1} t + \dots)$

G alg. group over $\mathbb{C}$

$Gr_{aff}(G) = G(F)/G(\mathcal{O})$

What is the affine Grassmanian for $GL_n(\mathbb{C})$, $SL_n(\mathbb{C})$, $PGL_n(\mathbb{C})$?

$Gr^{aff}_{GL_n} = GL_n(F)/GL_n(\mathcal{O})$

$GL_n(\mathcal{O})$ nxn matrices w/ coeff in $\mathcal{O}$ whose inverse is of the same sort.

$\begin{pmatrix} t & \\ & t \end{pmatrix} \in gl_2(\mathcal{O})$

$\begin{pmatrix} t^{-1} & \\ & t^{-1} \end{pmatrix} = \begin{pmatrix} t & \\ & t \end{pmatrix}^{-1}$

$\uparrow gl_2(F)$ here the inverse is not in $gl_2(\mathcal{O})$.

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Eg: $n=1$

$GL_1(F) = F^\times$ nonzero elts $F^\times = F - \{0\}$.

$GL_1(\mathcal{O}) = \mathcal{O}^\times = \{ p \in \mathcal{O} \mid \text{val}(p) = 0 \}$.

i.e. has $\neq 0$ constant term.

Affine Grassmanian $F^\times / \mathcal{O}^\times$

$f \in F^\times \quad f = t^{\text{val}(f)} p$ where p has val 0.

$p \in \mathcal{O}^\times$.

Get a complete set of coset reps. $\cdots t^{-2}, t^{-1}, t^0, t^1, t^2, \cdots$

$F^\times / \mathcal{O}^\times \simeq \mathbb{Z}$. exponents add.

Lattices $V = n$ dim. vector over F

F^n

A lattice is a maximal rank $\mathcal{O}$ submodule of V.

Let $v_1, \cdots, v_n$ be a basis for V. an F basis.

$L = \mathcal{O}v_1 + \cdots + \mathcal{O}v_n$ is a lattice.

e_1, e_2 standard basis for F^2

$\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$

$L_0 = \left\{ \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \middle| \; p_1, p_2 \in \mathcal{O} \right\}$.

$\mathcal{L} = \{ \text{Lattices in } V \} \simeq GL_n(F) / GL_n(\mathcal{O})$.

isom. as $GL_n(F)$ sets.

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$GL_n(F)$ acts in $\mathcal{L}$.

$L \subset V \qquad g(L) \subset V.$

$L = \mathcal{O}v_1 + \cdots + \mathcal{O}v_n$

$g(L) = \mathcal{O}g(v_1) + \cdots + \mathcal{O}g(v_n)$

Action is transitive. $\qquad V \simeq F^n$

$e_1, \ldots, e_n$

$g \circlearrowright \begin{matrix} L_0 = \mathcal{O}e_1 + \cdots + \mathcal{O}e_n \\ L = \mathcal{O}v_1 + \cdots + \mathcal{O}v_n. \end{matrix}$

$g = (v_1 | \cdots | v_n)$.

write v_i in terms of the basis $e_1, \ldots, e_n$ and write it as the ith col. of g.

invertible b/c takes a basis to a basis.

$g \in GL_n(F)$.

$\mathrm{Stab}(L_0) = \{ g \in GL_n(F) \mid g(L_0) = L_0 \}$.

$g(e_i) \in L_0. \quad \Rightarrow$ ith col. of g has all coeff in $\mathcal{O}$. $\forall i$.

$g^{-1} \in \mathrm{Stab}(L_0)$ same is true for g^{-1}.

$\Rightarrow g \in GL_n(\mathcal{O})$. converse is obv.

$\Rightarrow \mathcal{L} \simeq GL_n(F) / GL_n(\mathcal{O})$.

$n=1$ Aff Grass is $\mathbb{Z}$. ∞ dim as an inc. union of thing 0 dim.

$GL_n(F)/GL_n(\mathcal{O})$ ind-varieties

increasing union of algebraic varieties.

indexed by $\mathbb{N}$.

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Take a lattice L

Fact: $\exists m>0$ such that $t^{-m}L_0 \supset L \supset t^m L_0$.

L $v_1 \begin{pmatrix} t^{-5}+2t^{-4}+\dots \\ t^4+7t^8+\dots \end{pmatrix}$

$v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $n=5$ works.

$$X_m = \{ L \in \mathcal{L} \mid t^m L_0 \supset L \supset t^m L_0 \}.$$

$\mathcal{L} = \bigcup X_m$ increasing union.

Consider the f.d. complex vectorspace $t^m L_0 / t^m L_0 = W_m$

$$\{ t^i e_j + t^m L_0 \mid -m \le i < m,\ 1 \le j \le n \}$$

L is uniquely determined by $L / t^m L_0$ is a subset of W_m

$X_m \simeq \{$subspaces in $W_m\}$

$\|$

Grassmanian of W_m.

This is how we can put a topology on affine Grassmanian.

$SL_n(F) = \{ g \in gl_n(F) \mid \det g = 1 \}$.

$SL_n(F) / SL_n(\mathcal{O})$ $\mathcal{L}$ It is no longer true that $SL_n(F)$ acts transitively on L.

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We can talk about the validation of a lattice.

$L = \mathcal{O}v_1 + \cdots + \mathcal{O}v_n$

$\det(v_1 | v_2 | \cdots | v_n) \in F^\times$

↑ depends on the basis

$\mathrm{val}(\det(v_1 | \cdots | v_n))$ does not.

$\| \mathrm{Val}(L)$

$\mathcal{L}_k = \{\text{lattices of valuation } k\}$.

$SL_n(F)$ acts transitively on $\mathcal{L}_k$ for each k.

In particular, on $\mathcal{L}_0$.

$\mathcal{L}_0 \ni L = \mathcal{O}v_1 + \cdots + \mathcal{O}v_n$

$L_0 \in \mathcal{L}_0$

$g = (v_1 | \cdots | v_n) \qquad g(L_0) = L$

$\det g \in \mathcal{O}^\times$

$\| c$

change the basis $g' = (\frac{1}{c}v_1 | \cdots | v_n)$

$\det g' = 1$. so $g' \in SL_n(F)$.

$\mathrm{Stab}_{SL_n(F)}(L_0) = \mathrm{Stab}_{GL_n(F)}(L_0) \cap SL_n(F)$

$= GL_n(\mathcal{O}) \cap SL_n(F)$

$= SL_n(\mathcal{O}) = \{g \in GL_n(\mathcal{O}) \mid \det g = 1\}$.

$L_0 \simeq Gr$

$\simeq SL_n(F)/SL_n(\mathcal{O})$.

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affine flag manifold.

$GL_n(F)$

$\mathcal{LC} = \{ (L_i) \mid L_i \in \mathcal{L}_i \quad L_{i+n} = tL_i$

↑Lattice Chains $\quad L_{-1} \supset L_0 \supset L_1 \supset L_2 \supset \cdots$

↑lattice of val 0. (not std lattice).

$= \{ L_0 \supset L_1 \supset \cdots \supset L_{n-1} \supset tL_0 \}$.

$GL_n(F)$ acts transitively on $\mathcal{LC}$.

Eg: standard lattice chain

$L_0 = \langle e_1, \dots e_n \rangle$ notation for $\mathcal{O}$ span.

$\cup$

$L_1 = \langle e_1, \dots, e_{n-1}, te_n \rangle$

$\cup$

$L_2 = \langle e_1, \dots, te_{n-1}, te_n \rangle$

$\cup$

$\vdots$

$L_{n-1} = \langle e_1, te_2, \dots, te_n \rangle$

$\cup$

$tL_0 = \langle te_1, \dots, te_n \rangle$.

Stab(std. lattice chain) = ?

will do for 2x2 general case exercise.

$L_0 = \langle e_1, e_2 \rangle \qquad L_1/tL_0$ is a line in L_0/tL_0.

$L_1 = \langle e_1, te_2 \rangle$

$g \in \text{Stab}(L_0 \supset L_1 \supset tL_0) \qquad g(L_0) = L_0 \quad g(L_1) = L_1$

$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad ge_1 \in L_0 \quad ge_2 \in L_0.$

$ge_1 = \begin{pmatrix} a \\ c \end{pmatrix} \qquad ge_2 = \begin{pmatrix} b \\ d \end{pmatrix}$

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$\Rightarrow ae_1 + ce_2 \qquad a, c \in \mathcal{O}$

$$\mathrm{val}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \geq 0$$

$g(e_1) \in L_1 \qquad g(te_2) \in L_1$

$\| \qquad\qquad \|$

$\begin{pmatrix} a \\ c \end{pmatrix} \qquad \begin{pmatrix} tb \\ td \end{pmatrix}$

$\mathrm{val}\, a \geq 0 \qquad td \in t\mathcal{O} \qquad \mathrm{val}\, d \geq 0$

$\mathrm{val}\, c \geq 1 \qquad b \in \mathcal{O} \qquad \mathrm{val}\, b \geq -1.$

$b \in t^{-1}\mathcal{O}$

$$g = \begin{pmatrix} \mathrm{val} \geq 0 & \mathrm{val} \geq 0 \\ \mathrm{val} \geq 1 & \mathrm{val} \geq 0 \end{pmatrix}$$

$\mathrm{Stab} = I = \{ g \in GL_2(\mathcal{O}) \mid g \text{ is} \}.$

Iwahori

back to $GL_n(\mathcal{O})$

$\pi : GL_n(\mathcal{O}) \xrightarrow{t=0} GL_n(\mathbb{C})$

$$\begin{pmatrix} 1+2t & 7+8t\cdots \\ 2+3t & 1+4t+\cdots \end{pmatrix} \mapsto \begin{pmatrix} 1 & 7 \\ 2 & 1 \end{pmatrix}.$$

$GL_n(\mathbb{C}) \qquad GL_n(\mathcal{O}) \xrightarrow{\pi} GL_n(\mathbb{C})$

$\cup \qquad\qquad \cup \qquad\qquad \cup$

$B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \qquad I = \pi^{-1}(B) \qquad B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$

04/19/10.

Stab(std. lattice chain) = I

$GL_n(F)/I \simeq \mathcal{H}$.

CHRIS: Recall: convolution product on G/B
defined using $\mathcal{O}_i \tilde{\times} \mathcal{O}_j$
where $\mathcal{O}_i, \mathcal{O}_j \subset G/B$ are irreducible B-orbits.
in sense that B acts trans. on them.

Convolution on $G(F)/G(\mathcal{O})$
will involve $\mathcal{O}_i \tilde{\times} \mathcal{O}_j$ where $\mathcal{O}_i, \mathcal{O}_j \subset Gr_G^{aff}$
are $G(\mathcal{O})$-orbits.

Reminder: what are the $G(\mathcal{O})$ orbits in Gr_G^{aff}?

$G = GL_n$

dominant weight λ for GL_n is a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$
where $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$

$$L_\lambda = t^{\lambda_1} e_1 \mathcal{O} + t^{\lambda_2} e_2 \mathcal{O} + \dots + t^{\lambda_n} e_n \mathcal{O}$$

$$L_\lambda \longleftrightarrow \begin{pmatrix} t^{\lambda_1} & & & \\ & t^{\lambda_2} & & \\ & & \ddots & \\ & & & t^{\lambda_n} \end{pmatrix} \cdot G(\mathcal{O}).$$

Claim: $G(\mathcal{O})$ orbits in $Gr_{GL_n}^{aff}$ are precisely
the orbits of L_λ & orbits of L_λ's
are disjoint.

e.g. in GL_2 $L = gL_0$ $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

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Consider the $G(\mathcal{O})$-orbit of L.

e.g. $G(\mathcal{O})L = \{ h \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} G(\mathcal{O}) \mid h \in G(\mathcal{O}) \}$.

algorithm to show that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ lies in the $G(\mathcal{O})$-orbit of $\begin{pmatrix} t^{\lambda_1} & \\ & t^{\lambda_2} \end{pmatrix} \cdot G(\mathcal{O})$.

- Apply row/col. operations so that $\mathrm{val}(a) \leq \mathrm{val}(b)$, $\mathrm{val}(c)$, $\mathrm{val}(d)$
- Apply row/col operations to clear out b & c.

$$\begin{pmatrix} t^{n_1} p_1 & t^{n_2} p_2 \\ t^{n_3} p_3 & t^{n_4} p_4 \end{pmatrix} \qquad p_i \in \mathcal{O}^\times \text{ usual.} \quad n_1 \leq n_2, n_3, n_4$$

$$\begin{pmatrix} 1 & 0 \\ \frac{t^{n_3}}{t^{n_1}} \frac{p_3}{p_1} & 1 \end{pmatrix} \begin{pmatrix} t^{n_1} p_1 & t^{n_2} p_2 \\ t^{n_3} p_3 & t^{n_4} p_4 \end{pmatrix} = \begin{pmatrix} t^{n_1} p_1 & t^{n_2} p_2 \\ 0 & * \end{pmatrix}$$

$\frac{t^{n_3}}{t^{n_1}} \frac{p_3}{p_1} \in \mathcal{O}$ $\Big\{$

$$\begin{pmatrix} t^{n_1} p_1 & 0 \\ 0 & t^{n_4} p_4 \end{pmatrix}$$

last operation $\begin{pmatrix} p_1^{-1} & \\ & p_4^{-1} \end{pmatrix} \begin{pmatrix} & \\ & \end{pmatrix} = \begin{pmatrix} t^{n_1} & \\ & t^{n_4} \end{pmatrix}$

$\lambda = n_1 \geq n_4$ or mult by permutation matrix to put in the right order.

Question: $G(\mathcal{O})$ stabilizes $L_0 \Rightarrow$ stabilizes $t^{-m}L_0$ & $t^m L_0$.

$\Rightarrow$ $GL_n(\mathcal{O})$ acts the set of lattices $L : t^m L_0 \subseteq L \subset t^{-m} L$. $\Rightarrow$ $G(\mathcal{O})$-orbits

$\Rightarrow$ any $GL_n(\mathcal{O})$ orbit lies inside of $Gr(W_m)$.

Question: What is the closure of a $G(\mathcal{O})$ orbit in $Gr^{aff}_{GL_n}$?
& same question for $Gr^{aff}_{SL_2}$?

Week #12 — 04/21/10

$$\begin{pmatrix} t^a & \\ & t^{-a} \end{pmatrix}\begin{pmatrix} f_1 & f_2 \\ tf_3 & f_4 \end{pmatrix}\begin{pmatrix} t^{-b} & \\ & t^b \end{pmatrix} = \begin{pmatrix} t^{a-b}f_1 & f_2 t^{a+b} \\ t^{-a-b+1}f_3 & f_4 t^{b-a} \end{pmatrix} \in \mathcal{O}_{(a+b-1,\,-a-b)}$$

$\in \mathcal{O}_{(a,-a)}$ $\qquad$ $\in \mathcal{O}_{(b,-b)}$

Dominant Weight

sequence of integers $\lambda = \{\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \mid \lambda_i \in \mathbb{Z}\}$

at first allow $\lambda_n < 0$. (positive if $\lambda_n \geq 0$).

Positive dominant weights: λ such that $\lambda_i \in \mathbb{N} = \{0,1,2\dots\}$

length n dominant weights $\rightsquigarrow$ $GL_n(\mathcal{O})$-orbits in $Gr^{aff}(GL_n)$

$$\lambda \longmapsto GL_n(\mathcal{O})\begin{pmatrix} t^{-\lambda_1} & & \\ & t^{-\lambda_2} & \\ & & \ddots\, t^{-\lambda_n} \end{pmatrix} GL_n(\mathcal{O})/GL_n(\mathcal{O}) = \mathcal{O}_\lambda$$

Last time:
$\mathcal{O}_\lambda = G(\mathcal{O})\begin{pmatrix} t^{\lambda_1} & & \\ & t^{\lambda_2} & \\ & & \ddots\, t^{\lambda_n} \end{pmatrix} G(\mathcal{O})/G(\mathcal{O})$. Last time mistake.

"for which will have - signs for an easier proof"
was wrong convention.

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proof: that every orbit of the form $\mathcal{O}_\lambda$.
Gaussian elimination.

Define: $Gr^{\geq 0}(GL_n) = \{L \in Gr^{aff}(GL_n) \mid L \supset L^0\}$

Remark: If $L \supset L^0$ then there exists $n \geq 0$ such that $t^{-n} L \supset L^0$.

positive dominant weights correspond $GL_n(\mathcal{O})$-orbits in $Gr^{\geq 0}(GL_n)$.

$$\begin{pmatrix} t^{-\lambda_1} & & \\ & t^{-\lambda_2} & \\ & & \ddots \\ & & & t^{-\lambda_n} \end{pmatrix} \longleftrightarrow \text{span}\left(t^{-\lambda_1} e_1, t^{-\lambda_2} e_2, \ldots, t^{-\lambda_n} e_n\right) \supset L^0$$

Ordering on dominant weights

Def. $\gamma_i = (0,0,\ldots,0,-1,1,0,\ldots,0)$
(-1 in the ith position, 1 in the $i+1^{st}$)

$\lambda \leq \mu$ if $\lambda = \mu + \sum a_i \gamma_i$ $\quad a_i \geq 0$

e.g. $\lambda = (1,1)$ $\quad \mu = (2,0)$

$\lambda = (2,0) + (-1,1)$ $\quad \lambda \leq \mu$.

we can also have them noncomparable.

Theorem: in $Gr^{\geq 0}(GL_n)$

$\overline{\mathcal{O}_\mu} = \bigcup_{\lambda \leq \mu} \mathcal{O}_\lambda$ & $Gr^{\geq 0}$ is closed.

Question: What do $IC(\mathcal{O}_\mu)$ look like?

Supported on $\bigcup_{\lambda \leq \mu} \mathcal{O}_\lambda$.

$\{\lambda^1, \ldots, \lambda^m\} = \{\lambda \leq \mu\}$
$\lambda^m = \mu$
$N = \dim \mathcal{O}_\mu$.

Fill:

	$\mathcal{O}_{\lambda^1}$	$\mathcal{O}_{\lambda^2}$	$\mathcal{O}_{\lambda^3}$	...	$\mathcal{O}_{\lambda^m}$
⋮					
N					

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Coweights in SL_2

If $\begin{pmatrix} t^{\lambda_1} & \\ & t^{-\lambda_2} \end{pmatrix} \in SL_2(F)$

$-\lambda_2 = \lambda_1$

Dominant coweights for SL_2 are indexed by $2\mathbb{N}$ and correspond to $SL_2(\mathcal{O})$-orbits in $Gr^{aff}(SL_2)$

$$2n \longmapsto SL_2(\mathcal{O}) \cdot \begin{pmatrix} t^{-n} & \\ & t^{n} \end{pmatrix} \cdot SL_2(\mathcal{O}) \Big/ SL_2(\mathcal{O}) =: \mathcal{O}_{2n}$$

If $\overset{\lambda}{2m}, \overset{\mu}{2n}$ are SL_2 coweights, then $\lambda \leq \mu$ if $m \leq n$.
(Check this is equivalent to $(m_1, -m) = (n_1, -n_1) + q_i(r_i)$)

Fact: $\overline{\mathcal{O}_{2n}} = \bigcup_{0 \leq m \leq n} \mathcal{O}_{2m}$

Convolution Product on $Gr^{aff}(G)$

For us G will be GL_n or SL_2

$$\begin{array}{ccccc} G(F) \times G(F) & \xrightarrow{\ g\ } & G(F) \times G(F)/G(\mathcal{O}) & & \\ \downarrow p & & \downarrow m & \searrow \pi & \\ G(F)/G(\mathcal{O}) \times G(F)/G(\mathcal{O}) & & G(F)/G(\mathcal{O}) & & G(F)/G(\mathcal{O}) \end{array}$$

$\pi(g_1, g_2 G(\mathcal{O})) = g_2 G(\mathcal{O})$ (over $g_1 G(\mathcal{O})$)

$m(g_1, g_2 G(\mathcal{O})) = g_1 g_2 G(\mathcal{O})$

$g(g_1, g_2) = (g_1, g_2 G(\mathcal{O}))/\sim$

$p(g_1, g_2) = (g_1 G(\mathcal{O}), g_2 G(\mathcal{O}))$.

$(g_1 b, b^{-1} g_2 G(\mathcal{O})) \sim (g_1, g_2 G(\mathcal{O}))$

where $b \in G(\mathcal{O})$

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Def: $\mathcal{O}_\lambda, \mathcal{O}_\mu$ are 2 $G(\mathcal{O})$ orbits in $Gr^{aff}(G)$ indexed by weights λ and μ, then

$$\mathcal{O}_\lambda \tilde{\times} \mathcal{O}_\mu = \left\{ (g_1, g_2 G(\mathcal{O})) \in G(F) \times_{G(\mathcal{O})} G(F)/G(\mathcal{O}) \;\middle|\; \begin{array}{l} g_1 G(\mathcal{O}) \in \mathcal{O}_\lambda \\ \text{and} \\ g_2 G(\mathcal{O}) \in \mathcal{O}_\mu \end{array} \right\}$$

Question: What orbits lie in $m(\mathcal{O}_\lambda \tilde{\times} \mathcal{O}_\mu)$?

$$IC(\mathcal{O}_\lambda) \tilde{\boxtimes} IC(\mathcal{O}_\mu) = IC(\mathcal{O}_\lambda \tilde{\times} \mathcal{O}_\mu).$$

then

$$IC(\mathcal{O}_\lambda) * IC(\mathcal{O}_\mu) := m_*(IC(\mathcal{O}_\lambda) \tilde{\boxtimes} IC(\mathcal{O}_\mu)).$$

$m(\mathcal{O}_{2m} \tilde{\times} \mathcal{O}_{2n}) \cap \mathcal{O}_{2\ell} \neq \phi$ for $|n-m| \leq \ell \leq m+n$

example 1:

$$\left[\begin{pmatrix} t^{-n} & \\ & t^n \end{pmatrix}, \begin{pmatrix} t^{-m} & \\ & t^m \end{pmatrix} G(\mathcal{O}) \right] \in \mathcal{O}_{2n} \tilde{\times} \mathcal{O}_{2m}$$

m ↘

$$\begin{pmatrix} t^{-n-m} & \\ & t^{n+m} \end{pmatrix} G(\mathcal{O}) \in \mathcal{O}_{2(n+m)}$$

$$\left[\begin{pmatrix} 1 & -1 \\ & 1 \end{pmatrix} \begin{pmatrix} t^{-n} & \\ & t^n \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ & 1 \end{pmatrix} \begin{pmatrix} t^{-m} & \\ & t^m \end{pmatrix} G(\mathcal{O}) \right] \in \mathcal{O}_{2n} \tilde{\times} \mathcal{O}_{2m}$$

m ↘

$$\begin{pmatrix} t^{n-m} & \\ & t^{m-n} \end{pmatrix} \in \mathcal{O}_{|2(n-m)|}$$

$\in I$ 04/21/10

$$\left[\begin{pmatrix} -1 & \\ 1 & \end{pmatrix}\begin{pmatrix} t^{-n} & \\ & t^{n} \end{pmatrix}\cdot\begin{pmatrix} -1 & \\ 1 & \end{pmatrix}\begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix}\begin{pmatrix} t^{-m} & \\ & t^{m} \end{pmatrix} G(\mathcal{O})\right]$$

$\in \mathcal{O}_{2n} \tilde{\times} \mathcal{O}_{2m}$

$\downarrow m$

$$\begin{pmatrix} t^{n} & \\ & t^{-n} \end{pmatrix}\cdot\begin{pmatrix} t^{-m}f_1 & t^{m}f_2 \\ t^{-m+1}f_3 & t^{m}f_4 \end{pmatrix} G(\mathcal{O})$$

$$= \begin{pmatrix} t^{-n-m}f_1 & t^{m+n}f_2 \\ t^{-m-n+1}f_3 & t^{m-n}f_4 \end{pmatrix} G(\mathcal{O})$$

claim: $\in \mathcal{O}_{2(m+n-1)}$.

$$IC(\mathcal{O}_{2m}) \tilde{*} IC(\mathcal{O}_{2n})$$

$$\| $$

$$IC(\mathcal{O}_{2(m+n)}) \oplus IC(\mathcal{O}_{2(m+n-1)}) \oplus \cdots \oplus IC(\mathcal{O}_{2(m-n)}).$$

Week #13 — 04/28/10

Kimberly

$$C = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \quad X_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad X_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$C \to \begin{pmatrix} 2j & & \\ & 2j-2 & \\ & & \ddots \\ & & & -2j \end{pmatrix} \qquad X_+ \to \begin{pmatrix} 0 & \mu_1 & & \\ & \ddots & \ddots & \\ & & \ddots & \mu_r \\ & & & 0 \end{pmatrix} \qquad X_- \to \begin{pmatrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}$$

$$\mu_i = i(2j - i + 1)$$

$$D_j \otimes D_k = D_{j+k} \oplus D_{j+k-1} \oplus \cdots \oplus D_{|j-k|}$$

Facts: $C(v \otimes w) = C(v) \otimes w + v \otimes C(w)$

If v and w are both eigenvectors of C, then so is $v \otimes w$ and its eigenvalue is the sum of eigenvalues for v and w.

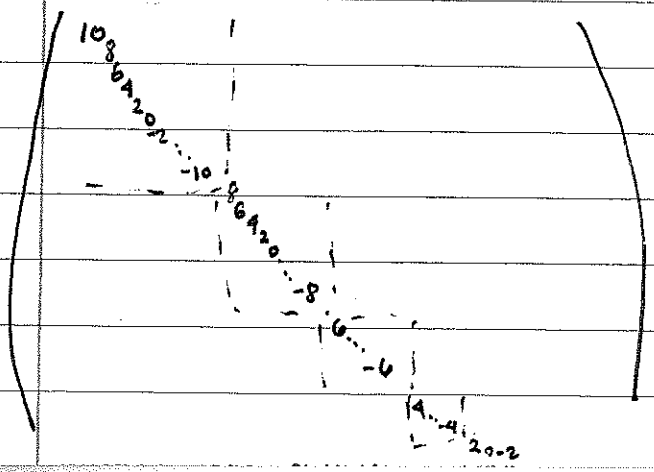
Consequently, the list of eigenvalues on $D_i \otimes D_j$ are pairwise sums of an eigenvalue from D_i with an eigenvalue from D_j.

Example: $D_3 \otimes D_2 = D_5 \oplus D_4 \oplus \cdots \oplus D_1$

$$D_3 = \begin{pmatrix} 6 & & & & \\ & 4 & & & \\ & & 2 & & \\ & & & 0 & \\ & & & & -2 \\ & & & & & -4 \\ & & & & & & -6 \end{pmatrix}; \quad D_2 = \begin{pmatrix} 4 & & & \\ & 2 & & \\ & & 0 & \\ & & & -2 \\ & & & & -4 \end{pmatrix}$$

evals: 10, 8, 6, 4, 2, 0, -2, -4, -6, -8, -10

mult: 1 2 3 4 5 5 5 4 3 2 1



04/28/10

Achar.

Weight: homomorphism

$$\begin{pmatrix} * & \\ & * \end{pmatrix} \longrightarrow \mathbb{C}^\times$$

$\subset SL_2(\mathbb{C})$

i.e. $\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} \longrightarrow \mathbb{C}^\times$

Every such homom. is of the form

$$\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} \longmapsto a^n \qquad n \in \mathbb{Z}.$$

Lie alg. version

$$\left\{ \begin{pmatrix} * & \\ & * \end{pmatrix} \right\} \longrightarrow \mathbb{C}$$

$\mathbb{C}[\varepsilon]$

In f.d. reps only get integer weights.

Coweight: it's a homom from $\mathbb{C}^\times \longrightarrow \begin{pmatrix} * & \\ & * \end{pmatrix} \in SL_2(\mathbb{C})$

any such homom is

$$t \longmapsto \begin{pmatrix} t^n & \\ & t^{-n} \end{pmatrix}$$

$\begin{pmatrix} t^n & \\ & t^{-n} \end{pmatrix}$ is either a map $\mathbb{C}^\times \longrightarrow SL_2(\mathbb{C})$

or an element of $SL_2(F)$

↑ Laurent Series.

$$\left\{ \begin{matrix} SL_2(\mathcal{O})\text{-orbits} \\ \text{on } Gr^{aff} \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{dominant} \\ \text{coweights} \end{matrix} \right\}$$

means $n \geq 0$.

Look at weights, coweights for $PGL_2(\mathbb{C}) = SL_2(\mathbb{C})/\pm I$

A weight of $PGL_2(\mathbb{C})$ is a map

$$\left\{ \begin{pmatrix} * & \\ & * \end{pmatrix} / \pm I \right\} \longrightarrow \mathbb{C}^\times$$

these correspond to the even weights of $SL_2(\mathbb{C})$.

Coweights for $PGL_2(\mathbb{C})$

$$\mathbb{C}^\times \longrightarrow \left\{ \begin{pmatrix} * & \\ & * \end{pmatrix} / \pm I \right\}$$

$$t \longmapsto \begin{pmatrix} t^{n/2} & \\ & t^{-n/2} \end{pmatrix}$$

sq. root match up.

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wts of SL_2 $= \mathcal{H} \supset 2\mathcal{H}$ = wts for $PGL_2(\mathbb{C})$

coweights of SL_2 $= \mathcal{H} \subset \frac{1}{2}\mathcal{H}$ = cowts for $PGL_2(\mathbb{C})$

$\mathcal{H}$ = $SL_2(\mathbb{O})$-orbits in Gr^{aff}

↑ Reparametrize by mult by 2

General notion: 'Langlands duality' (swaps weights and cowts)
- $SL_2(\mathbb{C})$ is Lang. dual to $PGL_2(\mathbb{C})$
- $GL_n(\mathbb{C})$ is self dual.

Geometric Satake Equivalence

(Perv. Sh. on $Gr^{aff}(SL_2)$) $\xleftrightarrow{\sim}$ Reps of $PGL_2(\mathbb{C})$ = Reps of $SL_2(\mathbb{C})$ or $\mathfrak{sl}_2(\mathbb{C})$ w/ even evals.

(get by composing with π)

$SL_2(\mathbb{C}) \xrightarrow{\pi} PGL_2(\mathbb{C})$

∩

{Perv. Sh. on $Gr^{aff}(PGL_2)$} $\longleftrightarrow$ Reps of $SL_2(\mathbb{C})$

Last week:

orbits in $Gr^{aff}(SL_2)$ labeled by even integers

$\mathcal{O}_0, \mathcal{O}_2, \mathcal{O}_4, \ldots$ $\quad \bar{\mathcal{O}}_n = \bigcup_{m \le n} \mathcal{O}_m$ $\quad$ Fact: $\dim \mathcal{O}_n = n$

Picture:

$Gr^{aff}(PGL_2)$: disconnected, 2 components

One looks like $Gr^{aff}(SL_2)$

other one: $\mathcal{O}_1$ $\mathcal{O}_3$ $\mathcal{O}_5$ $\mathcal{O}_7$ ...

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elts. correspond to lattice chains

$L \supset tL \supset t^2L \supset \cdots$

Fact: $IC(\mathcal{O}_n)$ is:

(n even)

	$\mathcal{O}_n$	$\mathcal{O}_{n-2}$	$\mathcal{O}_{n-4}$	$\cdots$	$\mathcal{O}_0$
0					
-1					
-2					
⋮					
$-n$	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}$	$\cdots$	$\mathbb{C}$

NOTE: Answer is different & more complicated if you replace $\mathbb{C}$ by something like $\overline{\mathbb{F}}_p$

Related: $\mathrm{Reps}(SL_2(\overline{\mathbb{F}}_p))$ is more complicated.

Know in advance: $m \geq n$

$$IC(\mathcal{O}_n) * IC(\mathcal{O}_m) \simeq IC(\mathcal{O}_{n+m}) \oplus \cdots \oplus IC(\mathcal{O}_{|n-m|})$$

Table for these perv. sh. (m, n both even). ($|n-m| = m-n$)

$m \geq n$

	$\mathcal{O}_{m+n}$	$\mathcal{O}_{m+n-2}$	$\mathcal{O}_{m+n-4}$	$\cdots$	$\mathcal{O}_{m-n}$	$\cdots$	$\mathcal{O}_0$
0							
-1							
-2				all $\mathbb{C}$'s here			
⋮							
$-m+n$			$\mathbb{C}$				
⋮							
$-m-n+2$	0	$\mathbb{C}$	$\mathbb{C}$	$\cdots$	$\mathbb{C}$	$\cdots$	$\mathbb{C}$
odd rows always 0 $-m-n+1$	0	0	0	$\cdots$	0	$\cdots$	0
$-m-n$	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}$	$\cdots$	$\mathbb{C}$	$\cdots$	$\mathbb{C}$

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Problem that remains:

Describe the spaces

$$m^{-1}(x) \cap (\Theta_p \tilde{\times} \Theta_q)$$

$x \in \Theta_k$

$$m(\Theta_p \tilde{\times} \Theta_q) = \Theta_{p+q} \cup \Theta_{p+q-2} \cup \dots \cup \Theta_{|p-q|}$$

so $m^{-1}(x) \cap (\Theta_p \tilde{\times} \Theta_q) = \phi$ if $k \notin \{p+q, p+q-2, \dots, |p-q|\}$.

$IC(\Theta_2) * IC(\Theta_2)$

	Θ_4	Θ_2	Θ_0
0			
-1	0	0	0
-2		$\mathbb{C}$	$\mathbb{C}$
-3	0	0	0
-4	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}$

twisted strata in the support of

$IC(\Theta_2) \tilde{\boxtimes} IC(\Theta_2)$

(x)
- $\Theta_2 \tilde{\times} \Theta_2$
- $\Theta_2 \times \Theta_0$
- $\Theta_0 \tilde{\times} \Theta_2$
- $\Theta_0 \tilde{\times} \Theta_0$

Find: $m^{-1}(x) \cap$ (each of those (x)) for $x \in \Theta_0, \Theta_2, \Theta_4$.

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k, p, q	$m^{-1}(x) \cap (\Theta_p \tilde{\times} \Theta_q)$; $x \in G_k$	
$0,0,0$	pt	
any $,0,0$	empty	
$0,2,0$	$x \in \Theta_0$ $m^{-1}(x) \cap \Theta_2 \tilde{\times} \Theta_0$ $\{2\}$.	Φ
$0,0,2$		Φ
$0,2,2$	?? Guess: $\mathbb{P}^1 \times C'$	Hard cases
$2,2,0$		
$2,0,2$		
$2,2,2$		
$4,2,2$	pt	
4, any	Φ	

Maria

Week #14

05/05/10

Jared: $IC(\theta_2) * IC(\theta_2)$

	θ_4	θ_2	θ_0
0			$\mathbb{C}$
-1			
-2		$\mathbb{C}$	$\mathbb{C}$
-3			
-4	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}$

	$\theta_0 \tilde{\times} \theta_0$	$\theta_2 \tilde{\times} \theta_2$	$\theta_2 \tilde{\times} \theta_0$	$\theta_2 \tilde{\times} \theta_2$: $X_2 \times X_2$	$X_2 \times Y_2$	$Y_2 \times X_2$	$Y_2 \times Y_2$
$X \in \theta_0$	pt	ϕ	ϕ	ϕ	$\mathbb{C}^2$	$\mathbb{C}^1$	ϕ
$X \in \theta_2$	ϕ	X_2: pt; Y_2: ϕ	X_2: pt; Y_2: ϕ	ϕ	$\mathbb{C}^X$	ϕ	ϕ
$X \in \theta_4$	ϕ	ϕ	ϕ	pt.			

$$IC(\theta_2) * IC(\theta_2) = m_*\left(IC(\theta_2) \tilde{\boxtimes} IC(\theta_2)\right)$$

supported on $\overline{\theta_2 \tilde{\times} \theta_2} = \theta_2 \tilde{\times} \theta_2 \cup \theta_0 \tilde{\times} \theta_2 \cup \theta_2 \tilde{\times} \theta_0 \cup \theta_0 \tilde{\times} \theta_0$.

need to compute $X \in \theta_0, \theta_2, \theta_4$.

$m^{-1}(X) \cap \theta_p \tilde{\times} \theta_q \qquad p, q \in \{0, 2\}$.

$\theta_{2n} = G(\theta)\begin{pmatrix} t^n & \\ & t^{-n} \end{pmatrix} G(\theta)$.

From last time: (Chris)

$$m(\theta_p \tilde{\times} \theta_q) = \theta_{p+q} \cup \theta_{p+q-2} \cup \cdots \cup \theta_{|p-q|}.$$

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$\Theta_0 = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} G(\theta)$

$\Theta_2 = G(\theta) \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} G(\theta)$

$\Theta_4 = G(\theta) \begin{pmatrix} t^2 & \\ & t^{-2} \end{pmatrix} G(\theta)$.

claim: $A, B \in G(\theta)$, then

$$(A + t^2 B) \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} G(\theta) = A \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} G(\theta).$$

proof: This is the same as:

$$\begin{pmatrix} t^{-1} & \\ & t \end{pmatrix} A^{-1} (A + t^2 B) \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \in G(\theta).$$

$$\| \quad \text{Id} + \begin{pmatrix} t^{-1} & \\ & t \end{pmatrix} A^{-1} t^2 B \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix}$$

$$\| \quad \text{Id} + \begin{pmatrix} 1 & \\ & t^2 \end{pmatrix} A^{-1} B \begin{pmatrix} t^2 & \\ & 1 \end{pmatrix} \in G(\theta). \quad //$$

$\Theta_2 = \left\{ \begin{pmatrix} \text{tm polys} \end{pmatrix} \right\} \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} G(\theta)$.

Facts: ① $\Theta_2 = I \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} G(\theta) \cup I \begin{pmatrix} t^{-1} & \\ & t \end{pmatrix} G(\theta)$

Iwahori: $I = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid t^{-1} c \in \Theta \right\}$.

$X_2 = I \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} G(\theta)$

$Y_2 = I \begin{pmatrix} t^{-1} & \\ & t \end{pmatrix} G(\theta)$.

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② $X_2 = I\begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} G(\theta)$

$$= \left\{ \begin{pmatrix} t & \alpha t^{-1}+\beta \\ & t^{-1} \end{pmatrix} \epsilon^{G(\theta)} \;\middle|\; \alpha, \beta \in \mathbb{C} \right\}.$$

Sketch: $A = \begin{pmatrix} a_0 + a_1 t & b_0 + b_1 t \\ c_1 t & d_0 + d_1 t \end{pmatrix} \in I$

$$A\begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} = \begin{pmatrix} a_0 t + a_1 t^2 & b_0 t^{-1} + b_1 \\ c_1 t^2 & d_0 t^{-1} + d_1 \end{pmatrix}$$

$$= \begin{pmatrix} t & \alpha t^{-1}+\beta \\ & t^{-1} \end{pmatrix} \begin{pmatrix} a_0 + a_1 t - \alpha c_1 t - \beta c_1 t^2 & -\beta d_1 \\ c_1 t^3 & d_0 + d_1 t \end{pmatrix}$$

$$\alpha = \frac{b_0}{d_0} \qquad \beta = \frac{b_1}{d_0} - \frac{b_0 d_1}{d_0^2}.$$

③ $Y_2 = I\begin{pmatrix} t^{-1} & \\ & t \end{pmatrix} G_1(\theta)$

$$= \left\{ \begin{pmatrix} t^{-1} & \\ \gamma & t \end{pmatrix} G(\theta) \;\middle|\; \gamma \in \mathbb{C} \right\}.$$

$X = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} G(\theta) \in \Theta_0$

$m^{-1}(X) \cap \Theta_0 \tilde{\times} \Theta_0 = \left(\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} G(\theta) \right) = p_t$

$m^{-1}(X) \cap X_2 \tilde{\times} Y_2 = \left\{ \left(\begin{pmatrix} t & \alpha t^{-1}+\beta \\ & t^{-1} \end{pmatrix}, \begin{pmatrix} t & a t^{-1}+b \\ & t^{-1} \end{pmatrix} G(\theta) \;\middle|\; \right.$

$\left. \begin{pmatrix} t & \alpha t^{-1}+\beta \\ & t^{-1} \end{pmatrix} \begin{pmatrix} t & a t^{-1}+b \\ & t^{-1} \end{pmatrix} G(\theta) = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} G(\theta) \right\}$

$$\begin{pmatrix} t & \alpha t^{-1}+\beta \\ & t^{-1} \end{pmatrix}\begin{pmatrix} t & a t^{-1}+b \\ & t^{-1} \end{pmatrix} \in G(\theta)$$

$$\begin{pmatrix} t^2 & \sim \\ & t^{-2} \end{pmatrix} \notin G(\theta)$$

so $m^{-1}(X) \cap X_2 \tilde{\times} X_2 = \phi$.

$$m^{-1}(X) \cap X_2 \tilde{\times} Y_2 = \left\{ \begin{pmatrix} t & \alpha t^{-1}+\beta \\ & t^{-1} \end{pmatrix}\begin{pmatrix} t & \\ \gamma & t^{-1} \end{pmatrix} \middle| \begin{pmatrix} 1+\alpha\gamma t^{-1}+\beta\gamma & \alpha+\beta t \\ \gamma t^{-1} & 1 \end{pmatrix} \in G(\theta) \right\}$$

$\| \mathbb{C}^2$

$\Rightarrow \gamma = 0$

α, β arbitrary.

$$m^{-1}(X) \cap Y_2 \tilde{\times} X_2 = \left\{ \begin{pmatrix} t^{-1} & \\ \gamma & t \end{pmatrix}, \begin{pmatrix} t & \alpha t^{-1}+\beta \\ & t^{-1} \end{pmatrix} G(\theta) \right|$$

$$\begin{pmatrix} t^{-1} & \\ \gamma & t \end{pmatrix}\begin{pmatrix} t & \alpha t^{-1}+\beta \\ & t^{-1} \end{pmatrix} G(\theta) = \begin{pmatrix} 1 & \\ & \end{pmatrix} G(\theta)$$

$$\begin{pmatrix} 1 & \alpha t^{-2}+\beta t^{-1} \\ \gamma t & \alpha\gamma t^{-1}+\gamma\beta+1 \end{pmatrix}$$

$X \in \theta_2 \qquad X = \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} G(\theta)$

$$m^{-1}(X) \cap X_2 \tilde{\times} X_2 = \left\{ \begin{pmatrix} t & \alpha t^{-1}+\beta \\ & t^{-1} \end{pmatrix}, \begin{pmatrix} t & a t^{-1}+\beta \\ & t^{-1} \end{pmatrix} G(\theta) \right|$$

$$\begin{pmatrix} t^2 & a+bt+\alpha t^2+\beta t^{-1} \\ & t^{-2} \end{pmatrix} G(\theta) = \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} G(\theta).$$

$$\begin{pmatrix} t^{-1} & \\ & t \end{pmatrix}\begin{pmatrix} t^2 & a+bt+\alpha t^{-2}+\beta t^{-1} \\ & t^{-2} \end{pmatrix} \in G(\theta)$$

$$\begin{pmatrix} t & a t^{-1}+b+\alpha t^{-3}+\beta t^{-2} \\ 0 & t^{-1} \end{pmatrix} \in G(\theta)$$

$m^{-1}(x) \cap X_2 \tilde{x} X_2 = \phi.$

$$m^{-1}(x) \cap X_2 \tilde{x} Y_2 = \left\{ \begin{pmatrix} t & \alpha t^{-1}+\beta \\ & t^{-1} \end{pmatrix}, \begin{pmatrix} t^{-1} & \\ \gamma & t \end{pmatrix} G(\theta) \;\middle|\; \begin{pmatrix} t & \alpha t^{-1}+\beta \\ & t^{-1} \end{pmatrix}\begin{pmatrix} t^{-1} & \\ \gamma & t \end{pmatrix} G(\theta) = \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} G(\theta) \right\}.$$

$$\begin{pmatrix} t^{-1} & \\ & t \end{pmatrix}\begin{pmatrix} 1+\alpha\gamma t^{-1}+\beta\gamma & \alpha+\beta t \\ \gamma t^{-1} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} t^{-1}+\alpha\gamma t^{-2}+\beta\gamma t^{-1} & \alpha t^{-1}+\beta \\ \gamma & t \end{pmatrix}$$

$\alpha = 0, \quad \beta\gamma = -1$

$m^{-1}(x) \cap X_2 \tilde{x} Y_2 = \mathbb{C}^\times$

$x = \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} G(\theta) \in \Theta_2$

$$m^{-1}(x) \cap \Theta_0 \tilde{x} \Theta_2 = \left\{ \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, A\begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} G(\theta) \;\middle|\; A\begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} G(\theta) = \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} G(\theta) \right\}.$$

$$\begin{pmatrix} t^{-1} & \\ & t \end{pmatrix}\begin{pmatrix} & \\ & \end{pmatrix}\begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \in G(\theta).$$

$$m^{-1}(x) \underset{\substack{\| \\ pt}}{\cap} \theta_0 \tilde{x} X_2 = \left\{ \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \begin{pmatrix} t & \alpha t^{-1}+\beta \\ & t^{-1} \end{pmatrix} G(\theta) \;\middle|\; \begin{pmatrix} t & \alpha t^{-1}+\beta \\ & t^{-1} \end{pmatrix} G(\theta) = \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} G(\theta) \right.$$

$$\begin{pmatrix} t^{-1} & \\ & t \end{pmatrix}\begin{pmatrix} t & \alpha t^{-1}+\beta \\ & t^{-1} \end{pmatrix} \in G(\theta)$$

$$\| $$

$$\begin{pmatrix} 1 & \alpha t^{-2}+\beta t^{-1} \\ & 1 \end{pmatrix} \qquad \alpha=\beta=0 \quad \Rightarrow pt.$$

$$m^{-1}(x) \underset{\substack{\| \\ \text{empty.}}}{\cap} \theta_0 \tilde{x} Y_2 = \left\{ \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \begin{pmatrix} t^{-1} & \\ \gamma & t \end{pmatrix} G(\theta) \;\middle|\; \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}\begin{pmatrix} t^{-1} & \\ \gamma & t^{-1} \end{pmatrix} G(\theta) = \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} G(\theta). \right.$$

$$\begin{pmatrix} t^{-1} & \\ & t \end{pmatrix}\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}\begin{pmatrix} t^{-1} & \\ \gamma & t \end{pmatrix} = \begin{pmatrix} t^{-2} & \\ & \cdots \end{pmatrix}$$

empty.