PARTITIONING INTO TWO GRAPHS WITH ONLY SMALL COMPONENTS

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ABSTRACT. The paper presents several results on edge partitions and vertex partitions of graphs into graphs with bounded size components. We show that every graph of bounded tree-width and bounded maximum degree admits such partitions. We also show that an arbitrary graph of maximum degree three has a vertex partition into two graphs, each of which has components on at most two vertices, and an edge partition into two graphs, each of which has components on at most eight vertices. It is not known whether similar results are true for maximum degree four and five, but we show that a similar result is false for maximum degree six or higher, even for planar graphs.

1. INTRODUCTION

Graphs in this paper are simple, that is, without loops or multiple edges. The set of vertices of a graph $G$ will be denoted by $V(G)$, and the set of edges of $G$ will be denoted by $E(G)$. An edge partition of a graph $G$ is a set $\{A_1, A_2, \ldots, A_k\}$ of subgraphs of $G$ such that $\bigcup_{i=1}^{k} E(A_i) = E(G)$. Similarly, a vertex partition of $G$ is a set $\{A_1, A_2, \ldots, A_k\}$ of induced subgraphs of $G$ such that $\bigcup_{i=1}^{k} V(A_i) = V(G)$.

Observe that vertex coloring and edge coloring are special cases of partitions. More precisely, a proper vertex $k$-coloring is a vertex partition into $k$ edgeless graphs, and a proper edge $k$-coloring is an edge partition into $k$ matchings. Of course, there are many results on proper coloring, but other types of partitions have been studied as well. Tutte [8] considered edge partitions of arbitrary graphs into planar graphs, Nash-Williams [7] considered edge partitions of arbitrary graphs into forests, while Chartrand and Kronk [2] considered vertex partitions of arbitrary graphs into forests. Further types of partitions can be found in [1], [4], and [5]. These results answer questions of the following type: Given classes of graphs $\mathcal{G}$, $\mathcal{S}_1$, $\mathcal{S}_2$, $\ldots$, $\mathcal{S}_k$, does every graph $G \in \mathcal{G}$ have a vertex partition (or edge partition) $\{G_1, G_2, \ldots, G_k\}$ such that each $G_i$ is in $\mathcal{S}_i$? Note that a negative answer to questions of the above type may be viewed as a Ramsey-theoretic result.

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Let $k$ be a positive integer. A $k$-tree is a graph defined inductively as follows: A complete graph on $k$ vertices is a $k$-tree. If $G$ is a $k$-tree, and $K$ is a subgraph of $G$ that is a complete graph on $k$ vertices, then a graph obtained from $G$ by adding a new vertex and joining it by new edges to all vertices of $K$ is a $k$-tree. Any subgraph of a $k$-tree is a partial $k$-tree. The tree-width of a graph $G$ is zero if $G$ is edgeless; otherwise it is the smallest integer $k$ such that $G$ is a partial $k$-tree. Nontrivial forests have tree-width 1, while every graph has some tree-width.

In a previous paper [4], the authors consider (among other problems) finding partitions of graphs embedded on surfaces into partial $k$-trees. Also, in [3], the authors find partitions of partial $k$-trees into graphs of even lower tree-width. The partitions in this paper will be different, and are a natural generalization of proper colorings. Note that a proper vertex (edge) coloring can be described as a vertex (edge) partition into graphs with only components of one vertex (at most two vertices). In this paper, we investigate the existence of vertex and edge partitions into two graphs with only components of bounded size.

As the first result of the paper, we show that graphs of bounded tree-width and bounded maximum degree admit an edge partition and a vertex partition into two graphs with bounded size components. We also investigate whether bounding just one of tree-width and maximum degree suffices to guarantee the existence such partitions. The answer turns out to be easy for tree-width, but it is much more difficult for maximum degree. We show that an arbitrary graph of maximum degree three has a vertex partition into two graphs, each of which has components on at most two vertices, and an edge partition into two graphs, each of which has components on at most eight vertices. It is not known whether similar results are true for maximum degree four and five, but we show that a similar result is false for maximum degree six or higher, even for planar graphs.

2. Bounding Both Tree-Width and Maximum Degree

The main theorem of this section is based on the following result [3]. A tree-partition of a graph $G$ is a pair $(T, P)$ where $T$ is a tree and $P$ is a (disjoint) partition $\{P_t : t \in V(T)\}$ of $V(G)$ such that, for every pair of adjacent vertices $u$ and $v$ of $G$, either they are both contained in the same $P_t$, or there are two adjacent vertices $s$ and $t$ of $T$ such that $u \in P_s$ and $v \in P_t$. The width of a tree-partition $(T, P)$ is the maximum size of a $P_t$.

**Proposition 2.1.** Every graph of maximum vertex degree $\Delta$ and tree-width $k$ admits a tree-partition of width at most $24k\Delta$.

As a consequence of Proposition 2.1, we show that graphs of bounded tree-width and bounded maximum vertex degree can be vertex partitioned and edge partitioned into graphs whose connected components have bounded size.

**Theorem 2.2.** Let $k$ and $\Delta$ be positive integers, and let $G$ be a graph whose tree-width is at most $k$ and whose maximum vertex degree is at most $\Delta$. Then $G$ admits a vertex partition $\{G_1, G_2\}$ such that every connected component of $G_1$ and $G_2$ has at most $24k\Delta$ vertices, and $G$ admits an edge partition $\{H_1, H_2\}$ such that every connected component of $H_1$ and $H_2$ has at most $24k\Delta(\Delta + 1)$ vertices.

**Proof.** By Proposition 2.1, $G$ has a tree-partition $(T, P)$ of width at most $24k\Delta$ where $P = \{P_t : t \in V(T)\}$. Since $T$ is a tree, it has a vertex partition $\{T_1, T_2\}$
such that neither $T_1$ nor $T_2$ has any edges. Let $G_i = \bigcup_{t \in V(T_i)} P_t$ for $i \in \{1, 2\}$. It is clear that \{G_1, G_2\} is as described in Theorem 2.2.

Now we shall construct the edge partition \{H_1, H_2\}. Begin by choosing an arbitrary vertex $t_0$ of $T$. For each vertex $t$ of $T$, let $h(t)$ denote the set of vertices $s$ of $T$ such that $s = t$ or $s$ is a neighbor of $t$ that is separated from $t_0$ by $t$. For each $t$, let $H(t)$ denote the subgraph of $G$ that is induced by the edges with one endpoint in $P_t$ and the other endpoint in $P_{h(t)}$ for some $s \in h(t)$. Now let $H_i = \bigcup_{t \in V(T_i)} H(t)$ for $i \in \{1, 2\}$. Since $P_i$ has at most $24k\Delta$ elements, each of which has at most $\Delta$ neighbors, the conclusion follows.

3. **Bounding Only Tree-Width**

It is natural to ask whether bounding just one of tree-width and maximum vertex degree suffices to ensure the existence of a vertex partition or an edge partition into two graphs with bounded size components. We show that, in general, the answer to this question is negative. However, we also show that all graphs whose maximum vertex degree does not exceed three do admit such partitions.

In the remainder of this section, we consider the easy case of bounding only the tree-width. Let $S_n$ be a star on $2n$ vertices, that is, a tree with $2n - 1$ edges, all incident with the same vertex. Let $F_n$ be a fan on $n^2 + n + 1$ vertices, that is, a graph obtained from a path on $n^2 + n$ vertices by adding a new vertex and joining it to all vertices of the path. Observe that, if $n$ is a positive integer, the tree-width of $S_n$ is one, and the tree-width of $F_n$ is two. Yet it is clear that for every edge partition \{G_1, G_2\} of $S_n$ each of $G_1$ and $G_2$ is a star, and at least one of them has more than $n$ vertices. Similarly, it is easy to show that, for every vertex partition \{G_1, G_2\} of $F_n$, at least one of $G_1$ and $G_2$ has a connected component with more than $n$ vertices. It is worth noting that the above examples have the smallest tree-width possible, for a graph of tree-width zero has no edges, and a graph of tree-width one is a forest, and hence has a vertex partition into two edgeless parts.

4. **Bounding Only Maximum Degree**

In this section, we consider bounding only the maximum vertex degree. A graph is a near-triangulation if it is a plane graph whose every face, except possibly for the infinite face, is a triangle. For a positive integer $n$, let $T_n$ be the graph whose vertices are the triples of nonnegative integers summing to $n$, with an edge connecting two triples if they agree in one coordinate and differ by one in each of the other two coordinates. The graph $T_n$ may be viewed as embedded in the plane whose equation in $\mathbb{R}^3$ is $x + y + z = n$ where the name of each vertex forms its coordinates, and edges are straight line segments. The graph $T_5$ is illustrated in Figure 1. It is clear that each $T_n$ is a near-triangulation with no vertices of degree exceeding six. The next theorem states that it is impossible to find a vertex partition or an edge partition of $T_n$ into two graphs neither of which has connected components with more than $n$ vertices.

**Theorem 4.1.** If \{G_1, G_2\} is a vertex partition or an edge partition of $T_n$, then at least one of $G_1$ and $G_2$ has a connected component with more than $n$ vertices.

Before addressing the proof of the theorem, we need a few definitions. Let $G$ be a near-triangulation and let $v_1$, $v_2$, and $v_3$ be three distinct vertices in the cycle $C$ that bounds the infinite face. Then $v_1$, $v_2$, and $v_3$ induce a partition of $C$ into
paths $P_1$, $P_2$, and $P_3$ such that, for each $i \in \{1, 2, 3\}$, $P_i$ avoids $v_i$ and has the other two members of $\{v_1, v_2, v_3\}$ as endvertices. A connector of $G$ with respect to $\{v_1, v_2, v_3\}$ is a connected subgraph $H$ of $G$ such that, for each $i \in \{1, 2, 3\}$, the set $V(H) \cap V(P_i)$ is not empty.

The part of Theorem 4.1 that speaks of vertex partitions follows immediately from the following two results of [6].

![Graph $T_5$](image)

**Figure 1: $T_5$**

**Proposition 4.2.** Let $G$ be a near-triangulation and let $v_1$, $v_2$, and $v_3$ be distinct vertices in the cycle bounding the infinite face of $G$. For every vertex partition $\{G_1, G_2\}$ of $G$ there is a connector $H$ of $G$ with respect to $\{v_1, v_2, v_3\}$ that is a subgraph of $G_1$ or of $G_2$.

**Proposition 4.3.** If $H$ is a connector of $T_n$ with respect to $(0, 0, n)$, $(0, n, 0)$, and $(n, 0, 0)$, then $H$ has more than $n$ vertices.

The part of Theorem 4.1 on edge partitions follows immediately from Proposition 4.3 and the edge version of Proposition 4.2, which is stated and proved below.

**Proposition 4.4.** Let $G$ be a near-triangulation and let $v_1$, $v_2$, and $v_3$ be distinct vertices in the cycle bounding the infinite face of $G$. For every edge partition $\{G_1, G_2\}$ of $G$ there is a connector $H$ of $G$ with respect to $\{v_1, v_2, v_3\}$ that is a subgraph of $G_1$ or of $G_2$.

**Proof.** We will apply Proposition 4.2 to a graph $G'$ obtained from $G$ in the following process. Let the vertex set of $G'$ be $V(G) \cup E(G)$ with two such vertices being joined by an edge if and only if one of them is an edge $e$ and the other is either a vertex of $G$ incident with $e$, or an edge of $G$ that shares a common vertex and a common finite face with $e$. Alternatively, $G'$ may be viewed as obtained from $G$ by subdividing each of its edges once, and adding new edges incident with the new vertices so that each of the finite faces of $G$ becomes subdivided into four triangular faces. For example, if $G = T_n$, then $G'$ is isomorphic to $T_{2n}$.

Note that each of $v_1$, $v_2$, and $v_3$ lies in the cycle bounding the infinite face of $G'$. For each $i \in \{1, 2\}$, let $V_i = V(G) \cup E(G_i)$ and let $G'_i$ be the subgraph of $G'$ induced
by $V_i$. Then $\{G'_1, G'_2\}$ is a vertex partition of $G'$. Upon applying Proposition 4.2 to $G'$, we conclude that there is a connector $H'$ of $G'$ with respect to $v_1, v_2$, and $v_3$ that is a subgraph of $G'_1$ or of $G'_2$. Without loss of generality, we may assume that $H'$ is a connected component of $G'_1$. Let $H$ be a subgraph of $G$ induced by those vertices of $H'$ that are also vertices of $G$. We shall prove that $H$ is a connector of $G$ with respect to $v_1, v_2$, and $v_3$. Let $P_1, P_2,$ and $P_3$ be the paths that partition the cycle bounding the infinite face of $G$ as described in the definition immediately preceding Proposition 4.2, and let $P'_1, P'_2,$ and $P'_3$ be the corresponding paths in $G'$. Then, for each $i \in \{1, 2, 3\}$, the vertex set of $P'_i$ is the union of $V(P_i)$ and $E(P_i)$.

Suppose $H$ avoids one of the paths $P_i$ for some $i \in \{1, 2, 3\}$. But $H'$, being a connector, has a vertex $e$ in $P'_i$, which must be an edge of $P_i$. Let $v$ be a vertex that is incident in $G$ with $e$. Then, clearly, $v \in V(P_i)$. Since all vertices of $G$ are in $V_1$, which induces $G'_1$, and $H'$ is a connected component of $G'_1$, we conclude that $v$ is a vertex of $H'$, and hence also of $P_i \cap H$; a contradiction.

It remains to show that $H$ is connected. Let $u$ and $v$ be two vertices of $H$. Then, as $H'$ is connected, it contains a path $P$ from $u$ to $v$. Take the list of consecutive vertices of $P$ and modify it as follows: Between every two consecutive vertices of $P$ that are both edges of $G$ insert the vertex of $G$ that is incident with both edges. Since $V_1$ contains all vertices of $G$, and $G'_1$ is induced by $V_1$, the modified list consists of vertices of $H'$. The same list, when interpreted in $G$, alternates vertices and edges with two consecutive entries being incident. Since all vertices of $G$ that appear in the list are in $H'$, and hence in $H$, the list forms a walk in $H$ that begins in $u$ and ends in $v$. It follows that $H$ is a connector $G$ with respect to $v_1, v_2$, and $v_3$, as required.

The last theorem of the paper deals with graphs whose vertices have degree at most three.

**Theorem 4.5.** Let $G$ be a graph with no vertices of degree exceeding three. Then

(i) the graph $G$ has a vertex partition $\{G_1, G_2\}$ such that every connected component of $G_1$ and of $G_2$ has at most two vertices; and

(ii) the graph $G$ has an edge partition $\{H_1, H_2\}$ such that every connected component of $H_1$ and of $H_2$ is a path with at most eight vertices.

**Proof.** The proof of (i) is straightforward: From among all vertex partitions of $G$ choose one $\{G_1, G_2\}$ for which the number of edges of $G$ whose endvertices lie in different parts of the partition is maximum. Let $v$ be a vertex of $G_i$ for some $i \in \{1, 2\}$. The degree of $v$ in $G_i$ is at most one, since otherwise the partition of $G$ obtained from $\{G_1, G_2\}$ by moving $v$ together with the edges between $v$ and $V(G_3 - i)$, from $G_i$ to $G_3 - i$, contradict the choice of $\{G_1, G_2\}$. Thus $\{G_1, G_2\}$ is as described in (i).

The proof of (ii) is fairly complicated, and it occupies nearly all of the remainder of the paper. To aid in reading the proof, certain important points in the proof have been marked by labels (1)-(15). To construct $\{H_1, H_2\}$ as described in (ii), first we observe that $G$ may be assumed 3-regular, since every graph whose maximum vertex degree is at most three is a subgraph of a 3-regular graph $G$. The graph $G$ can be constructed by taking three disjoint copies of $G$, and for each vertex $v$ of $G$ whose degree $d$ is less than three, adding new $3 - d$ vertices and joining each of these new vertices to the three vertices of $G$ that correspond to $v$. 
For a subgraph $H$ of $G$, let $c(H)$ denote the number of cycles of $H$. Observe that a well-known theorem of Vizing [9] implies that $G$, having no vertex of degree exceeding three, can be properly edge-colored with at most four colors. This easily implies that $G$ has an edge partition $\{N_1, N_2\}$ such that each of $N_1$ and $N_2$ has maximum vertex degree at most two. From among all such edge partitions choose one $\{L_1, L_2\}$ for which $c(L_1) + c(L_2)$ is the minimum.

Suppose $c(L_1) + c(L_2) > 0$. Without loss of generality, we may assume that $L_1$ has a cycle. Let $e$ and $f$ be two adjacent edges of this cycle. Since $L_2$ has maximum degree two, from among the three vertices incident with $e$ and $f$ in $L_1$, at most two lie in the same connected component of $L_2$. Thus, by symmetry, we may assume that no path of $L_2$ joins the endpoints of $e$. Then $c(L_1 \setminus e) < c(L_1)$, while $c(L_2 \cup e) = c(L_2)$ and each of $L_1 \setminus e$ and $L_2 \cup e$ has no vertices of degree exceeding two. This contradiction to the choice of $\{L_1, L_2\}$ proves that all connected components of $L_1$ and of $L_2$ are paths.

Let $\mathcal{C}(G)$ denote the collection of connected components of a graph $G$. For a collection $\mathcal{S}$ of subgraphs of $G$, let $\omega(\mathcal{S})$ denote the number of elements of $\mathcal{S}$ on exactly $i$ edges. The weight of $\mathcal{S}$, denoted by $\omega(\mathcal{S})$, is the triple $(\omega(\mathcal{S}), \omega(\mathcal{S}), \omega(\mathcal{S}))$. The weight of an edge partition $\{G_1, G_2\}$ of $G$ is $\omega(\mathcal{C}(G_1) \cup \mathcal{C}(G_2))$. We order the weights lexicographically. From among all edge partitions $\{G_1, G_2\}$ of $G$ such that all components of $G_1$ and all components of $G_2$ are paths, let $\{L_1, L_2\}$ be one with minimum weight. We shall show that no path of $L_1$ and no path of $L_2$ has more than seven edges.

Let $\mathcal{S}$ be the set of those members of $\mathcal{C}(L_1) \cup \mathcal{C}(L_2)$ that have at least three edges. An edge $e$ of $G$ is free if it is an edge of an element $S$ of $\mathcal{S}$ and is incident with a vertex whose degree in $S$ is one. Let $F$ denote the subgraph of $G$ containing all of its vertices and all of its free edges. Observe that every free edge is adjacent to an edge that is not free. Thus

(1) no vertex of $F$ has degree exceeding two.

Suppose $S$ is a path of $L_1$ that has more than seven edges, and let $v$ be a vertex of $S$ whose distance in $S$ from each of its endvertices exceeds three. Let $f_0$ and $h_0$ be the edges of $S$ incident with $v$, and let $u$ and $w$ be the endvertices of, respectively, $f_0$ and $h_0$ that are different from $v$. For each $x \in \{u, w\}$, let $F_x$ be the component of $F$ that meets $x$. Clearly, the degree of $x$ in $F_x$ is at most one. This, together with (1), implies that each $F_x$ is a path. Let $y_x = x$ if $F_x$ has no edges, otherwise let $y_x$ be the endvertex of $F_x$ other than $x$.

We will show that $y_x$ is an endvertex of a path $T_x$ from $(\mathcal{C}(L_1) \cup \mathcal{C}(L_2)) \setminus \mathcal{S}$. Consider the case for $u$ (the $w$ case is symmetric). If $F_u$ is edgeless, then let $n = 0$. In this case the edge incident with $u = y_u$ not on $S$ is not free and must then be an edge of such a path. Otherwise, let $f_1, f_2, \ldots, f_n$ be the list of edges of $F_u$ so that for all $i \in \{1, 2, \ldots, n\}$, $f_i$ and $f_{i-1}$ are incident with the same vertex $v_i$. By induction, one can see that each $f_i$ is an endedge of a path $P_i$ in $\mathcal{S}$ with endvertex $v_i$. In this case the edge incident with $y_u$ not on $P_u$ is not free and must then be an edge of an appropriate path.

Since the degree of $v$ in $S$ is two and $G$ is 3-regular, at most one of $T_u$ and $T_w$ meets $v$. By symmetry, we may assume that

(2) $T_u$ avoids $v$.

Since $u$ is not an endvertex of $S$, it is incident with an edge $g_0$ of $S$ that is different from $f_0$. For each $i \in \{1, 2, \ldots, n\}$, let $g_i$ denote the edge that is incident with $f_i$
and is in the same component of $S$. Note that every vertex $y$ of $F_u$ different from $y_u$ avoids $T_u$ since it is incident with $f_i, g_i$, and $f_{i+1}$ for some $i \in \{0, 1, \ldots, n - 1\}$. Therefore,

(3) $T_u$ and $F_u$ meet in exactly one vertex, $y_u$.

Let now $L_i = L_i \Delta \{f_0, f_1, \ldots, f_n\}$ where $i \in \{1, 2\}$ and $\Delta$ denotes the symmetric difference operator. It is clear that $\{L_1, L_2\}$ is an edge partition of $G$. Also observe first that, by the definition of $F_u$,

(4) for every $i \in \{1, 2, \ldots, n\}$, the edges $f_{i-1}$ and $f_i$ are in different parts of $\{L_1, L_2\}$, and in different parts of $\{L_1', L_2'\}$, and

(5) for every $i \in \{0, 1, \ldots, n\}$, the edges $f_i$ and $g_i$ are in the same part of $\{L_1, L_2\}$, and in different parts of $\{L_1', L_2'\}$.

We shall show that

(6) each component of $L_1'$ and of $L_2'$ is a path, and

(7) the weight of $\{L_1', L_2'\}$ is smaller than the weight of $\{L_1, L_2\}$, thereby arriving at a contradiction.

Suppose first that (6) fails. Then at least one of $L_1'$ and $L_2'$ contains a cycle or a vertex of degree exceeding two. Assume first that a vertex $z$ has degree three in one of $L_1'$ and $L_2'$. It follows from (4) and the 3-regularity of $G$ that $z \in \{v, y_u\}$. But $z \neq v$ since $f_0$ and $h_0$ are in different parts of $\{L_1, L_2\}$, and $z \neq y_u$ since, by (5), $f_n$ and $g_n$ are in different parts of $\{L_1', L_2'\}$; a contradiction. Thus we may assume that a cycle $C$ is contained in one of $L_1'$ and $L_2'$. Since neither $L_1$ nor $L_2$ has cycles, $C$ contains $f_i$ for some $i \in \{0, 1, \ldots, n\}$. But then $i = n$ since otherwise one of the endvertices of $f_i$ is incident with $g_i$ and $f_{i+1}$, neither of which, by (4) and (5), is in the same part of $\{L_1', L_2'\}$ as $f_i$. It follows that $T_u \subseteq C$ since, by (5), $f_n$ and $g_n$ are in different parts of $\{L_1', L_2'\}$. Let $t$ denote the endvertex of $T_u$ other than $y_u$. By (3), $t$ does not lie on $F_u$, and, by (2) $t \neq v$. Hence the degree of $t$ in $C$ is one: a contradiction, which proves (6).

It remains to show that (7) holds. Let $T'_u$ denote the element of $\mathcal{C}(L_1') \cup \mathcal{C}(L_2')$ containing the edge incident with $v$ but not contained in $S$. For every $i \in \{0, 1, \ldots, n\}$, let $G_i$ denote the element of $\mathcal{C}(L_1) \cup \mathcal{C}(L_2)$ that contains $g_i$, and let $G'_i$ denote the element of $\mathcal{C}(L_1') \cup \mathcal{C}(L_2')$ that contains $g_i$. Let $H'_0$ be the element of $\mathcal{C}(L_1') \cup \mathcal{C}(L_2')$ that contains $h_0$. Clearly, $T_u$ is contained in some element of $\mathcal{C}(L_1') \cup \mathcal{C}(L_2')$. Denote this element by $T'_u$. Let $T'_v$ denote the element of $\mathcal{C}(L_1') \cup \mathcal{C}(L_2')$ that contains the edge of $T_u$ incident with $v$.

Let $\mathcal{L}$ denote the set of elements of $\mathcal{C}(L_1) \cup \mathcal{C}(L_2)$ that contain a vertex incident with an element of $\{f_0, f_1, \ldots, f_n\}$, and, similarly, let $\mathcal{L}'$ denote the set of elements of $\mathcal{C}(L_1') \cup \mathcal{C}(L_2')$ that contain a vertex incident with an element of $\{f_0, f_1, \ldots, f_n\}$. Then $\mathcal{L} = \{T_u, T'_v, G_0, G_1, \ldots, G_n\}$, and $\mathcal{L}' = \{T'_u, T'_v, H'_0, G'_0, G'_1, \ldots, G'_n\}$. Note that the elements of $\mathcal{L}$ and of $\mathcal{L}'$ listed here need not be distinct.

Now observe that

(8) \[ (\mathcal{C}(L_1) \cup \mathcal{C}(L_2)) \setminus \mathcal{L} = (\mathcal{C}(L_1') \cup \mathcal{C}(L_2')) \setminus \mathcal{L}'. \]

Thus, to show (7), it suffices to prove that

(9) \[ w(\mathcal{L}') < w(\mathcal{L}). \]

Let $I = \{i : 1 \leq i \leq n - 2 \}$ and $G_i \not\subseteq \{S, T_v\}$. Let $M = \{G_i : i \in I\}$ and let $M' = \{G'_i : i \in I\}$. For each $i \in I$, let $\sigma(i) = i$ if $|G_i \cap F_u| = 1$; otherwise let $\sigma(i) = j$ where $f_j \in G_i$ and $i \neq j$. Note that, for all $i \in I$, $G_i = G_j$ if and only if
$j \in \{i, \sigma(i)\}$, which holds if and only if $G'_i = G'_j$. Also, for all $i \in I$, $|G_i| = |G'_i|$ since $G'_i$ is obtained from $G_i$ by removing \{f_i, f_{\sigma(i)}\} and adding \{f_{i+1}, f_{\sigma(i+1)}\}. Thus

(10) the map $G_i \mapsto G'_i$ is a bijection from $M$ to $M'$, and $|G_i| = |G'_i|$ for all $i \in I$.

Let $N = \mathcal{L} \setminus M = \{T_u, T_v, G_0, G_{n-1}, G_n\}$ and, similarly, let $N' = \mathcal{L}' \setminus M' = \{T'_u, T'_v, H_0, G'_0, G'_{n-1}, G'_n\}$ keeping in mind that the elements of $N$ and of $N'$ listed above need not be distinct, and $G_{n-1}$ and $G'_{n-1}$ do not exist if $n = 0$. From (10) we conclude that, to show (9), it suffices to show that

(11) $w(N') < w(N)$.

To achieve this, we need to consider the cardinalities of, and the identifications between, the paths listed above in $N$ and $N'$. Let $S_1$ (respectively $S_2$) be the component of $S \setminus f_0$ that contains $g_0$ (respectively $h_0$). By the choice of $v$,

(12) $S_1$ has at least three elements, and $S_2$ has at least four elements.

Also, as shown earlier,

(13) $T_u$ has one or two edges.

Suppose first that $n = 0$. In this case, $H'_0 = S_2, G'_0 = G'_1 = S_1, T'_u = T'_v = T_u \cup T_v \cup \{f_0\}$ and $G'_{n-1}$ does not exist. Therefore all paths in $N'$ have at least three edges, whereas, by (13), $T_u$ has at most two edges, and, consequently, (11) holds.

Suppose now that $n$ is a positive integer. For a subgraph $A$ of $G$, let $\delta(A) = |E(A) \cap \{f_i\}|$. Note that $f_i$ is in at most one of $S_1, S_2$, and $T_v$. Also, $f_n$ is an edge of $G_n$, but not of $T_u \cup G_n$. Considering all ways in which the paths listed in the definition of $N$ and of $N'$ and path $f_n$ can coincide or intersect, we conclude that

\[
\begin{align*}
|E(G'_0)| &= |E(S_1)| + 1 - \delta(S_1) \\
|E(H'_0)| &= |E(S_2)| - \delta(S_2) \\
|E(T'_u)| &= |E(T_u)| + 1 - \delta(T_u).
\end{align*}
\]

(14)

Now, $T'_u = G'_{n-1}$ and, whether or not $G'_{n-1} = S$,

(15) each of $T'_u$ and $G'_{n-1}$ has at least four edges.

From (12)–(15) we conclude that $T_u$ has a most two edges, $T'_v$ has as least as many edges as $T_v$, and each of $G'_0, H'_0, T'_u$, and $G'_{n-1}$ has at least three edges. Thus (11) can fail only if $G'_n$ has fewer than three edges. In this case $G_n \notin \{S, T_v\}$ and $\delta(S_1) = \delta(S_2) = \delta(T_v) = 0$ so that each of $H'_0$ and $G'_0$ has at least four edges, $G_n$ has exactly three edges, and $T'_v$ has more edges than $T_v$. It follows easily that (11) holds. This also completes the proof of (9), and, thus, the proof of (7), and, consequently, the proof of Theorem 4.5.

An easy counting argument shows that if \{L_1, L_2\} is an edge partition of a cubic graph $G$ such that $\mathcal{C}(L_1) \cup \mathcal{C}(L_2)$ is a set of paths, then the average number of edges in these paths is three. If every path in $\mathcal{C}(L_1) \cup \mathcal{C}(L_2)$ has length exactly three, then by coloring the endedges of elements in $\mathcal{C}(L_1)$ red, coloring all endedges of elements in $\mathcal{C}(L_2)$ blue, and coloring all remaining edges green, we obtain a proper coloring of edges of $G$. Since not all cubic graphs admit such a coloring (e.g. the Petersen graph), the bound of seven in Theorem 4.5 cannot be reduced to three. This gives rise to the following:
Question 4.6. For $k \in \{4, 5, 6\}$, does every cubic graph $G$ have an edge partition 
$\{L_1, L_2\}$ such that each element of $C(L_1) \cup C(L_2)$ is a path with at most $k$ edges?

We have shown in Theorem 4.5 that every graph with maximum degree less than 
four has a vertex partition and an edge partition into two graphs, each of which has 
only small components. In Theorem 4.1, we have shown that, the existence of such 
decompositions cannot be guaranteed for graphs whose maximum degree exceeds 
five. These results leave the following question, which we cannot answer.

Question 4.7. Is there a number $n$, such that every graph with maximum degree 
4 or 5 has a vertex partition and an edge partition into two graphs such that each 
part has only components with at most $n$ vertices?

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