Causal Symmetric Spaces
Geometry and Harmonic Analysis

Joachim Hilgert       Gestur Ólafsson
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Preface

In the late 1970s several mathematicians independently started to study a possible interplay of order and continuous symmetry. The motivation for doing so came from various sources. K.H. Hofmann and J.D. Lawson [67] tried to incorporate ideas from geometric control theory into a systematic Lie theory for semigroups, S. Paneitz [147, 148] built on concepts from cosmology as propagated by his teacher I.E. Segal [157], E.B. Vinberg’s [166] starting point was automorphism groups of cones, and G.I. Ol’shanskii [137, 138, 139] was lead to semigroups and orders by his studies of unitary representations of certain infinite-dimensional Lie groups. It was Ol’shanskii who first considered the subject proper of the present book, causal symmetric spaces, and showed how they could play an important role in harmonic analysis. In particular, he exhibited the role of semigroups in the Gelfand-Gindikin program [34], which is designed to realize families of similar unitary representations simultaneously in a unified geometric way. This line of research attracted other researchers such as R.J. Stanton [159], B. Ørsted, and the authors of the present book [159].

This book grew out of the Habilitationsschrift of G. Ólafsson [129], in which many of the results announced by Ol’shanskii were proven and a classification of invariant causal structures on symmetric spaces was given. The theory of causal symmetric spaces has seen a rapid development in the last decade, with important contributions in particular by J. Faraut [24, 25, 26, 28] and K.-H. Neeb [114, 115, 116, 117, 120]. Its role in the study of unitarizable highest-weight representations is becoming increasingly clear [16, 60, 61, 88] and harmonic analysis on these spaces turned out to be very rich with interesting applications to the study of integral equations [31, 53, 58] and groupoid $C^*$-algebras [55]. Present research also deals with the relation to Jordan algebras [59, 86] and convexity properties of gradient flows [49, 125]. Thus it is not possible to write a definitive treatment of ordered symmetric spaces at this point. On the other hand, even results considered “standard” by the specialists in the field so far have either not appeared in print at all or else can be found only in the original
literature.
This book is meant to introduce researchers and graduate students with
a solid background in Lie theory to the theory of causal symmetric spaces,
to make the basic results and their proofs available, and to describe some
important lines of research in the field. It has gone through various stages
and quite a few people helped us through their comments and corrections,
encouragement, and criticism. Many thanks to W. Bertram, F. Betten, J.
Faraut, S. Helgason, T. Kobayashi, J. Kockmann, Kh. Koufany, B. Krötz,
K.-H. Neeb, and B. Ørsted. We would also like to thank the Mittag-Leffler
Institute in Djursholm, Sweden, for the hospitality during our stay there in
spring 1996.

Djursholm
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Introduction

Symmetric spaces are manifolds with additional structure. In particular they are homogeneous, i.e., they admit a transitive Lie group action. The study of causality in general relativity naturally leads to “orderings” of manifolds [149, 157]. The basic idea is to fix a convex cone (modeled after the light cone in relativity) in each tangent space and to say that a point \( x \) in the manifold precedes a point \( y \) if \( x \) can be connected to \( y \) by a curve whose derivative lies in the respective cone wherever it exists (i.e., the derivative is a timelike vector). Various technical problems arise from this concept. First of all, the resulting “order” relation need not be antisymmetric. This describes phenomena such as time traveling and leads to the concept of causal orientation (or quasi-order), in which antisymmetry is not required. Moreover, the relation may not be closed and may depend on the choice of the class of curves admitted. Geometric control theory has developed tools to deal with such questions, but things simplify considerably when one assumes that the field of cones is invariant under the Lie group acting transitively. Then a single cone will completely determine the whole field and it is no longer necessary to consider questions such as continuity or differentiability of a field of cones. In addition, one now has an algebraic object coming with the relation: If one fixes a base point \( o \), the set of points preceded by \( o \) may be viewed as a positive domain in the symmetric space and the set of group elements mapping the positive domain into itself is a semigroup, which is a very effective tool in studying causal orientations. Since we consider only homogeneous manifolds, we restrict our attention to this simplified approach to causality.

Not every cone in the tangent space at \( o \) of a symmetric space \( \mathcal{M} \) leads to a causal orientation. It has to be invariant under the action of the stabilizer group \( H \) of \( o \). At the moment one is nowhere near a complete classification of the cones satisfying this condition, but in the case of irreducible semisimple symmetric spaces it is possible to single out the ones which admit such cones. These spaces are then simply called causal symmetric spaces, and it turns out that the existence of a causal orientation puts severe restrictions
on the structure of the space. More precisely, associated to each symmetric space $\mathcal{M} = G/H$ there is an involution $\tau: G \to G$ whose infinitesimal version (also denoted by $\tau$) yields an eigenspace decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$ of the Lie algebra $\mathfrak{g}$ of $G$, where $\mathfrak{h}$, the Lie algebra of $H$, is the eigenspace for the eigenvalue 1 and $\mathfrak{q}$ is the eigenspace for the eigenvalue $-1$. Then the tangent space of $\mathcal{M}$ at $o$ can be identified with $\mathfrak{q}$ and causal orientations are in one-to-one correspondence with $H$-invariant cones in $\mathfrak{q}$.

A heavy use of the available structural information makes it possible to classify all regular (i.e., containing no lines but interior points) $H$-invariant cones in $\mathfrak{q}$. This classification is done in terms of the intersection with a Cartan subspace $\mathfrak{a}$ of $\mathfrak{q}$. The resulting cones in $\mathfrak{a}$ can then be described explicitly via the machinery of root systems and Weyl groups.

It turns out that causal symmetric spaces come in two families. If $\mathfrak{g} = \mathfrak{t} + \mathfrak{p}$ is a Cartan decomposition compatible with $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$, i.e., $\tau$ and the Cartan involution commute, then a regular $H$-invariant cone in $\mathfrak{q}$ either has interior points contained in $\mathfrak{t}$ or in $\mathfrak{p}$. Accordingly the resulting causal orientation is called compactly causal or noncompactly causal. Compactly causal and noncompactly causal symmetric spaces show a radically different behavior. So, for instance, noncompactly causal orientations are partial orders with compact order intervals, whereas compactly causal orientations need neither be antisymmetric nor, if they are, have compact order intervals. On the other hand, there is a duality between compactly causal and noncompactly causal symmetric spaces which on the infinitesimal level can be described by the correspondence $\mathfrak{h} + i\mathfrak{q} \leftrightarrow \mathfrak{t} + \mathfrak{iq}$. There are symmetric spaces which admit compactly as well as noncompactly causal orientations. They are called spaces of Cayley type and are in certain respects the spaces most accessible to explicit analysis.

The geometry of causal symmetric spaces is closely related to the geometry of Hermitian symmetric domains. In fact, for compactly causal symmetric spaces the associated Riemannian symmetric space $G/K$, where $K$ is the analytic subgroup of $G$ corresponding to $\mathfrak{t}$, is a Hermitian symmetric domain and the space $H/(H \cap K)$ can be realized as a bounded real domain by a real analog of the Harish–Chandra embedding theorem. This indicates that concepts such as strongly orthogonal roots that can be applied successfully in the context of Hermitian symmetric domains are also important for causal symmetric spaces. Similar things could be said about Euclidian Jordan algebras.

Harmonic analysis on causal symmetric spaces differs from harmonic analysis on Riemannian symmetric spaces in various respects. So, for instance, the stabilizer group of the basepoint is noncompact, which accounts for considerable difficulties in the definition and analysis of spherical functions. Moreover, useful decompositions such as the Iwasawa decomposition
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... do not have global analogs. On the other hand, the specific structural information one has for causal symmetric space makes it possible to create tools that are not available in the context of Riemannian or general semisimple symmetric spaces. Examples of such tools are the order compactification of noncompact causal symmetric spaces and the various semigroups associated to a causal orientation.

The applications of causal symmetric spaces in analysis, most notably spherical functions, highest-weight representations, and Wiener-Hopf operators, have not yet found a definitive form, so we decided to give a mere outline of the analysis explaining in which way the geometry of causal orientations enters. In addition we provide a guide to the original literature as we know it.

We describe the contents of the book in a little more detail. In Chapter 1 we review some basic structure theory for symmetric spaces. In particular, we introduce the duality constructions which will play an important role in the theory of causal symmetric spaces. The core of the chapter consists of a detailed study of the $H^*$- and $\mathfrak{h}$-module structures on $\mathfrak{q}$. The resulting information plays a decisive role in determining symmetric spaces admitting $G$-invariant causal structures. Two classes of examples are treated in some detail to illustrate the theory: hyperboloids and symmetric spaces obtained via duality from tube domains.

In Chapter 2 we review some basic facts about convex cones and give precise definitions of the objects necessary to study causal orientations. These definitions are illustrated by a series of examples that will be important later on in the text. The central results are a series of theorems due to Kostant, Paneitz, and Vinberg, giving conditions for finite-dimensional representations to contain convex cones invariant under the group action. We also introduce the causal compactification of an ordered homogeneous space, a construction that plays a role in the analysis on such spaces.

Chapter 3 is devoted to the determination of all irreducible symmetric spaces which admit causal structures. The strategy is to characterize the existence of causal structures in terms of the module structure of $\mathfrak{q}$ and then use the results of Chapter 1 to narrow down the scope of the theory to a point where a classification is possible. We give a list of symmetric pairs $(\mathfrak{g}, \mathfrak{h})$ which come from causal symmetric spaces and give necessary and sufficient conditions for the various covering spaces to be causal.

In Chapter 4 we determine all the cones that lead to causal structures on the symmetric spaces described in Chapter 3. Using duality it suffices to do that for the noncompact $G$ causal symmetric spaces. It turns out that up to sign one always has a minimal and a maximal cone giving rise to a causal structure. The cones are determined by their intersection with a certain (small) abelian subspace $\mathfrak{a}$ of $\mathfrak{g}$ and it is possible to characterize...
the cones that occur as intersections with $a$. Since the cone in $q$ can be recovered from the cone in $a$, this gives an effective classification of causal structures. A technical result, needed to carry out this program but of independent interest, is the linear convexity theorem, which describes the image of certain coadjoint orbits under the orthogonal projection from $q$ to $a$.

Chapter 5 is a collection of global geometric results on noncompact causal symmetric spaces which are frequently used in the harmonic analysis of such spaces. In particular, it is shown that the order on such spaces has compact intervals. Moreover, the causal semigroup associated naturally to the maximal causal structure is characterized as the subsemigroup of $G$ which leaves a certain open domain in a flag manifold of $G$ invariant. The detailed information on the ordering obtained by this characterization allows to prove a nonlinear analog of the convexity theorem which plays an important role in the study of spherical functions. Moreover, one has an Iwasawa-like decomposition for the causal semigroup which makes it possible to describe the positive cone of the symmetric space purely in terms of the “solvable part” of the semigroup, which can be embedded in a semigroup of affine selfmaps. This point of view allows us to compactify the solvable semigroup and in this way makes a better understanding of the causal compactification possible.

In Chapter 6 we pursue the study of the order compactification of non-compact causal symmetric spaces. The results on the solvable part of the causal semigroup together with realization of the causal semigroup as a compression semigroup acting on a flag manifold yield a new description of the compactification of the positive cone which then can be used to give a very explicit picture of the $G$-orbit structure of the order compactification.

The last three chapters are devoted to applications of the theory in harmonic analysis.

In Chapter 7 we sketch a few of the connections the theory of causal symmetric spaces has with unitary representation theory. It turns out that unitary highest-weight representations are characterized by the fact that they admit analytic extensions to semigroups of the type considered here. This opens the way to construct Hardy spaces and gives a conceptual interpretation of the holomorphic discrete series for compactly causal symmetric spaces along the lines of the Gelfand–Gindikin program.

Chapter 8 contains a brief description of the spherical Laplace transform for noncompact causal symmetric spaces. We introduce the corresponding spherical functions, describe their asymptotic behavior, and give an inversion formula.

In Chapter 9 we briefly explain how the causal compactification from Chapter 2 is used in the study of Wiener–Hopf operators on noncompactly
causal symmetric spaces. In particular, we show how the results of Chapter
6 yield structural information on the $C^*$-algebra generated by the Wiener-
Hopf operators.

Appendix A consists of background material on reductive Lie groups and
their finite-dimensional representations. In particular, there is a collection
of our version of the standard semisimple notation in Section A.1. More-
over, in Section A.3 we assemble material on Hermitian Lie groups and
Hermitian symmetric spaces which is used throughout the text.

In Appendix B we describe some topological properties of the set of
closed subsets of a locally compact space. This material is needed to study
compactifications of homogeneous ordered spaces.