ASYMPTOTICS OF THE D’ALEMBERTIAN WITH POTENTIAL ON A PSEUDO-RIEMANNIAN MANIFOLD

THOMAS BRANSON AND GESTUR ÓLAFSSON

Abstract. Let □ be the Laplace-d’Alembert operator on a pseudo-Riemannian manifold \((M, g)\). We derive a series expansion for the fundamental solution \(G(x, y)\) of \(\square + H, H \in \mathcal{C}^{\infty}(M)\), which behaves well under various symmetric space dualities. The qualitative properties of this expansion were used in our paper in *Invent. Math.* 129 (1997) 63–74, to show that the property of vanishing logarithmic term for \(G(x, y)\) is preserved under these dualities.

1. Introduction. Let \((M, g)\) be a pseudo-Riemannian manifold of dimension \(n\), with Laplace–d’Alembert operator \(\square\). Let \(H\) be a smooth function on \(M\), and consider the problem of constructing a fundamental solution for the d’Alembertian with potential \(\square + H\).

A classical construction of Haïmmar formally develops the fundamental solution of \(\square + H\) in the form

\[
G(x, y, \square + H) = \begin{cases}
U(x, y)\sigma^{(2-n)/2} + V(x, y) \log |\sigma|, & n \text{ even}, \\
U(x, y)|\sigma|^{(2-n)/2}, & n \text{ odd},
\end{cases}
\]

for \((x, y)\) in a neighborhood \(\mathcal{O}\) of the diagonal in \(M \times M\), where \(\sigma = \sigma(x, y)\) is the geodesic distance-square, an \(U\) an \(V\) are smooth functions on \(\mathcal{O}\). Of course, the distance-square may be negative, since the metric may be indefinite. The true fundamental solution is a version of the formal expression (1) which is regularize, either in a classical sense [H], or in a distributional sense [C,F].

The precise analytic considerations needed to produce the fundamental solution vary according to the metric signature. This subject of this paper is a classical development of the series (1) which, for certain purposes, is a valuable alternative to the Haïmmar development. In particular, we use the existence of this development in an essential way in [BÓ]. There we prove that the vanishing of the logarithmic term \(V\) is preserved under various symmetric space dualities. This, in turn, allows us to construct many new locally symmetric spaces on which \(V\) vanishes, taking the potential \(H\) to be a constant multiple of the scalar curvature. The property of vanishing logarithmic term has been studied by many authors; for metrics of Lorentz signature, it is equivalent to Huygens’ principle; for Riemannian signature, there are interpretations with consequences for classical gravitation and electrostatics. In this paper, we limit ourselves to signature-independent considerations and develop...
the require \( j \) expansion algebraically, without treating convergence questions. In particular, all computations take place off \( \{ \sigma = 0 \} \).

Note that in this generality, the construction also produces formal asymptotic expansions of the resolvent kernels of same operators \( \Box + H \); i.e., of the kernel functions for the \( (\Box + H - \lambda)^{-1} \), where \( \lambda \in \mathbb{C} \).

In normal coordinates with origin at \( y \), let \( (x^\alpha) \) be the normal coordinates of the moving point \( x \). The coefficients \( V_k \) of the Taylor series for \( V \),

\[
V(x, y) \sim V_0(y) + (V_1)_\alpha(y) x^\alpha + (V_2)_{\alpha\beta}(y) x^{\alpha} x^{\beta} + \ldots
\]

\[
+ (V_k)_{\alpha_1 \ldots \alpha_k}(y) x^{\alpha_1} \ldots x^{\alpha_k} + \ldots,
\]

are universal local invariants of the metric \( g \) and \( j \) the potential \( H \), value \( j \) in the symmetric \( k \)-tensor field \( j \)'s on \( M \). (Here an \( j \) below, the summation convention is in force.) Similar considerations hold \( f \) for the function \( U \). In fact, the Taylor coefficients for \( U \) and \( V \) may be calculated \( j \) inductively an \( j \) algebraically from the Taylor expansions of the metric \( g \) and \( j \) the potential \( H \).

Choose normal coordinates for \( g \) in which \( y = 0 \) and \( x = (x^\alpha) \). It will be convenient to introduce an artificial flat reference metric: Let \( \eta \) be the standard flat metric of the same signature as \( g \). Then \( g \) has a normal coordinate expansion

\[
g_{\alpha\beta}(x) = \eta_{\alpha\beta} + g_{\alpha\beta,\gamma_1 \ldots \gamma_k}(y) x^{\gamma_1} \ldots x^{\gamma_k} + O(|x|^{k+1}),
\]

\[
g^{\alpha\beta}(x) = \eta^{\alpha\beta} + g^{\alpha\beta,\gamma_1 \ldots \gamma_k}(y) x^{\gamma_1} \ldots x^{\gamma_k} + O(|x|^{k+1}),
\]

where the \( | \cdot | \) in \( O(|x|^{k+1}) \) refers to any positive \( k \)-finite metric. The tensors \( g_{\alpha\beta,\gamma_1 \ldots \gamma_k} \) and \( g^{\alpha\beta,\gamma_1 \ldots \gamma_k} \) are local invariants; that is, they are linear combinations of monomials

\[
C(g \otimes \ldots \otimes g \otimes g^\# \otimes \ldots \otimes g^\# \otimes (\nabla \ldots \nabla R) \otimes \ldots \otimes (\nabla \ldots \nabla R)),
\]

where \( g^\# = (g^{\alpha\beta}) \) is the inverse of the metric \( g = (g_{\alpha\beta}) \). \( \nabla \) is the metric connection, \( R \) is the Riemann curvature tensor of \( \nabla \), and \( C \) is a contraction operator. As a consequence of the metric expansion (2), the normal coordinate metric determinant \( g \) has a similar expansion:

\[
g(x) = \pm 1 + g_{\alpha\beta}(y) x^\alpha x^\beta + \ldots + g_{\alpha_1 \ldots \alpha_k}(y) x^{\alpha_1} \ldots x^{\alpha_k} + O(|x|^{k+1}),
\]

where the tensors \( g_{\alpha\beta}(y) \) and \( g_{\alpha_1 \ldots \alpha_k}(y) \) are linear combinations of monomials of the form (3).

By Weyl’s invariant theory \( [W] \), the local invariants \( V_k \) are linear combinations of monomials of the form

\[
C(g \otimes \ldots \otimes g \otimes g^\# \otimes \ldots \otimes g^\# \otimes (\nabla \ldots \nabla R) \otimes \ldots \otimes (\nabla \ldots \nabla R)
\]

\[
\otimes (\nabla \ldots \nabla H) \otimes \ldots \otimes (\nabla \ldots \nabla H)).
\]

By taking note of the behavior of all terms under uniform dilatation \( g' = sg, 0 < s \in \mathbb{R} \), it is easy to compute that each monomial (4) in \( V_k \) (resp. \( U_j \)) enjoys a homogeneity property:

\[
p_{\nabla} + 2(p_R + p_H) = n - 2 + k \quad (\text{resp. } p_{\nabla} + 2(p_R + p_H) = j),
\]

where \( p_{\nabla} \) (resp. \( p_R, p_H \)) is the number of \( \nabla \) (resp. \( R, H \)) appearing. Note that implementation of the Ricci identities may convert occurrences of \( \nabla \) into occurrences of \( R \), but \( j \) does not \( j \) disturb the quantity \( p_{\nabla} + 2(p_R + p_H) \).
2. The expansion. Thus far, it has not been necessary to fix sign conventions on \( R \) an \( \square \) so now. \( \square \) is the total contraction of \( -g^\gamma \otimes \nabla \nabla f \). In a local frame \( (X_\alpha) \),
\[
R^\alpha_{\beta\gamma\delta}X_\alpha = R(X_\gamma, X_\delta)X_\beta .
\]
In local coordinates, the \( \square\) A\( \)l\( e\)mber\( t\)\( i\)\( a\)n is
\[
\square = -|g|^{-1/2} \partial_\beta (g^{\alpha\beta} |g|^{1/2} \partial_\alpha ) =: -\eta^{\alpha\beta} \partial_\alpha \partial_\beta + q^{\alpha\beta}(x) \partial_\alpha \partial_\beta + b^\alpha(x) \partial_\alpha .
\]
In normal coordinates, the vanishing of the first or \( \square \) er terms in the metric expansions gives
\[
q^{\alpha\beta}(x) = O(|x|^2), \quad b^\alpha(x) = O(|x|).
\]
Let \( D = -\eta^{\alpha\beta} \partial_\alpha \partial_\beta \). Our strategy will be to explicitly compute the effect of \( D \) on homogeneous terms of the expansion (1), an \( \square \) to qualitatively observe the effect of \( \square + H - D \). We break up a homogeneous term according to
\[
(V_k)_{\alpha_1...\alpha_k} x^{\alpha_1}...x^{\alpha_k} = V_k(x^{(k)}) = \sum_{\ell=0}^{[k/2]} \sigma^\ell V_k,\ell(x^{(k-2\ell)}),
\]
where \( x^{(k)} \) is the \( k \)-tuple (of \( n \)-tuples) \( (x, \ldots, x) \), an \( \square \) the \( V_k,\ell \) are trace free tensors. (That is, any contraction of \( V_k,\ell \) with \( \eta^\# \) vanishes.) In representation theoretic terms, we can accomplish this by taking the symmetric tensor representation of \( GL(n, \mathbb{R}) \) an \( \square \) decomposing into irreducible representations of the subgroup \( O(p, q) \), where \( (p, q) \) is the signature of \( \eta \). We similarly \( \square \) decompose each homogeneous term in the expansion of \( U \).

For simplicity, we restrict to \( \{ \sigma > 0 \} \) for purposes of this calculation. After insertion of appropriate minus signs, it is clear that the situation on \( \{ \sigma < 0 \} \) is also describe.

\( D \) obeys the second "or \( \square \) er Leibniz rule \( D(\varphi \psi) = \varphi D \psi + \psi D \varphi - 2 \eta^\#(d \varphi, d \psi) \). But if \( \Omega \) is an arbitrary trace free symmetric \( p \) tensor an \( \square \) \( \psi = \Omega(x^{(p)}) \),
\[
D(\psi) = \eta^\#(d \sigma, d \psi) = 2p \psi.
\]
Furthermore, \( D \sigma = -2n \) an \( \square \) \( \eta^\#(d \sigma, d \sigma) = 4 \sigma \), so that for \( s \in \mathbb{C} \),
\[
D(\sigma^s) = -2s(n + 2s - 2) \sigma^{s-1},
\]
\[
D(\sigma^s \log \sigma) = -2s^s - 1 \{ s(n + 2s - 2)(\log \sigma) + n + 4s - 2 \}.
\]
Putting all this together, we get (for \( \psi \) as above)
\[
D(\sigma^s \psi) = -2s(n + 2s + 2p - 2) \sigma^{s-1} \psi,
\]
\[
D(\sigma^s (\log \sigma) \psi) = -2s(n + 2s + 2p - 2) \sigma^{s-1} (\log \sigma) \psi - 2(n + 4s + 2p - 2) \sigma^{s-1} \psi .
\]
Thus for any \( t \in \mathbb{C} \), we have
\[
D(\sigma^t U_{j,\ell}(x^{(j-2\ell)})) = -2(\ell - t)(n - 2\ell - 2t + 2j - 2) \sigma^{t-1} U_{j,\ell}(x^{(j-2\ell)}),
\]
\[
D(\sigma^t (\log \sigma) V_{k,\ell}(x^{(k-2\ell)})) = \{-2\ell(n - 2\ell + 2k - 2) \sigma^{t-1} \log \sigma - 2(n + 2k - 2) \sigma^{t-1} \} V_{k,\ell}(x^{(k-2\ell)}).
\]
In particular, setting \( t = m := (n - 2)/2 \), we have

\[
D(\sigma^{\ell-m}U_{j,\ell}(x^{j-2\ell})) = -4(\ell - m)(j - \ell)\sigma^{\ell-m-1}U_{j,\ell}(x^{j-2\ell}),
\]

\[
D(\sigma^2 \log \sigma) V_{k,\ell}(x^{k-2\ell})) = -4\sigma^{\ell-1} \{\ell(k - \ell + m)(\log \sigma) + k + m\} V_{k,\ell}(x^{k-2\ell}).
\]

The important qualitative point about \( \Box + H - D \) is (6). We have:

**Lemma 1.** On \( \{ \sigma > 0 \} \), for formal power series \( U \) and \( V \) as in (7),

\[
(\Box + H)(\sigma^{-m}U + (\log \sigma)V) = \\
+ \sigma^{-m-1} \sum_{j=1}^{[j/2]} \left \{ W_j - \sum_{\ell=0}^{[j/2]} 4(\ell - m)(j - \ell)\sigma^\ell U_{j,\ell}(x^{j-2\ell}) \right \} \\
- \sigma^{-1} \sum_{k=0}^{[k/2]} \sum_{\ell=0}^{[k/2]} 4(k + m)\sigma^\ell V_{k,\ell}(x^{k-2\ell}) \\
+ \sigma^{-1} (\log \sigma) \sum_{k=0}^{[k/2]} \left \{ W_k - \sum_{\ell=0}^{[k/2]} 4\ell(k - \ell + m)\sigma^\ell V_{k,\ell}(x^{k-2\ell}) \right \},
\]

where \( W_j \) (resp. \( W_k \)) is a homogeneous polynomial of degree \( j \) (resp. \( k \)). \( W_{j,\ell} \) depends on \( \{U_u\} \) and \( \{V_v\} \) only through \( \{U_{u,L} \mid u < j, \ L \leq \ell \} \) and \( \{V_{v,L} \mid v < j-m, \ L \leq \ell \} \). \( W_{k,\ell} \) depends on \( \{U_u\} \) and \( \{V_v\} \) only through \( \{V_{v,L} \mid v < k, \ L \leq \ell \} \).

**Theorem 2.** If \( n \) is odd, given a constant \( U_0 \), the formula of Lemma 1, with \( V = 0 \), inductively computes a unique formal power series solution to the equation \( (\Box + H)(U|\sigma|^{-m}) = 0 \) on \( \{ \sigma \neq 0 \} \). The coefficients \( U_{j,\ell}(y) \) are universal local invariants as in (4). If \( n \) is even, the formula of Lemma 1(b) inductively computes a formal power series solution to \( (\Box + H)(\sigma^{-m} + V \log |\sigma|) = 0 \) on \( \{ \sigma \neq 0 \} \). This solution is unique modulo solutions without singularity at \( \sigma = 0 \). With the side conditions \( U_{j,m} = 0 \) for all \( j \), the even-dimensional solution is unique, and all coefficients \( U_{j,\ell}(y) \) and \( V_{k,\ell}(y) \) are universal local invariants as in (4).

**Proof.** First restrict to \( \{ \sigma > 0 \} \). Proceeding inductively, we fin \( \tilde{y} \) by examining the logarithm free terms that the \( U_{j,\ell} \) are uniquely \( \text{determine} \) for \( n \) odd. If \( n \) is even,

1. The \( U_{j,\ell} \) are uniquely \( \text{determine} \) for \( \ell < m \) and all \( j \);
2. The \( U_{j,m} \) may be prescribe \( \text{arbitrarily, but the} \ V_{k,0} \text{are uniquely} \text{determine} \text{for all} \ k \).

Switching attention to the terms with a \( \log \sigma \) factor, we get no additional condition on the \( V_{k,0} \), an \( \tilde{y} \)

3. The \( V_{k,\ell} \) are uniquely \( \text{determine} \) for \( \ell > 0 \) and all \( k \).

Going back to the logarithm free terms,

4. The \( U_{j,\ell} \) for \( \ell > m \) are uniquely \( \text{determine} \), given our prescription of the \( U_{j,m} \).

If \( E \) and \( F \) are two power series construct \( \tilde{y} \) as above, differing in the prescription of the \( U_{j,m} \) (an \( \tilde{y} \) its effects on the computation of the \( U_{j,\ell} \) for \( \ell > m \)), then \( E - F \)
is a power series (without singularity at $\sigma = 0$) satisfying $(\Box + H)(E - F) = 0$. Thus the power series construction is unique up to the a\,n\!\!\i a\,t\!\!\i on of such harmonics.

It is straightforward to insert signs as appropriate to extend the conclusion to $\{\sigma < 0\}$. The local invariance properties follow inductively from those of the metric, through (5).

The first term on the right in Lemma 1(a) regularizes to a constant multiple of the $\delta$-function when we take account of behavior through $\{\sigma = 0\}$, an\!\!\i com\!\!\i pute in the sense of distributions. Thus we are computing the fundamental solution of $\Box + H$ by implementing the above procedure. Since $\Box + H - \lambda$ is an operator of the same type, our results also cover the resolvent kernel, i.e., the kernel function of $(\Box + H - \lambda)^{-1}$.

3. Comparison with the Hadamard expansion. Some remarks on the relation of the above results to the Hadamard expansion are in order. In Hadamard's original treatment, $U(x, y)$ and $V(x, y)$ are expanded in series of two point functions:

$$U(x, y) \sim \sum_{j=0}^{\infty} K_j(x,y)\sigma^j, \quad V(x, y) \sim \sum_{k=0}^{\infty} L_k(x,y)\sigma^k.$$  

The $K_j$ and $L_k$ are determined, in a neighborhood of the diagonal in $M \times M$, by recursive solution of the transport equations; these are ordinary differential equations in which the geodesic parameter is the moving parameter.

With $x$ as the moving point and $y$ the fixed point, the definition of $\Box$ gives the following variants of (9). Let $G := \log |g|$ and $r = \sigma^{1/2}$, an\!\!\i t a\,n\!\!\i let a prime denote $d/dr$. Then

$$\Box \sigma^s = -4s(m + s + \frac{1}{2}rG')\sigma^{s-1},$$

$$\Box (\sigma^s \log \sigma) = -2s^{s-1}\{s(n + 2s - 2 + \frac{1}{2}rG')(\log \sigma) + n + 4s - 2 + \frac{1}{2}rG'\}.$$  

This gives

$$(\Box + H)(K_j \sigma^{j-m}) = ((\Box + H)K_j)\sigma^{j-m} - 4(j - m)\sigma^{j-m-1}(rK'_j + \{j + \frac{1}{2}rG'\}K_j),$$

$$(\Box + H)(L_k \sigma^k \log \sigma) = ((\Box + H)L_k)\sigma^k \log \sigma - 2k\sigma^{k-1} \log \sigma(2rL'_k + \{n + 2k - 2 + \frac{1}{2}rG'\}L_k) - 2\sigma^{k-1}(2rL'_k + \{n + 4k - 2 + \frac{1}{2}rG'\}L_k).$$

The analysis of, e.g., $[C,F]$ shows that, after suitable regularization and normalization of $U_0$,

$$(\Box + H)(\sigma^{-m}U + (\log \sigma)V) = \delta_y(x) + \text{(smooth)}.$$  

We need to make sure that $(\Box + H)(\sigma^{-m}U + (\log \sigma)V)$ vanishes to infinite order for $x \neq y$. The bottom $(\sigma^{-m-1})$ coefficient gives the condition

$$rK'_0 + \frac{1}{2}rG'K_0 = 0;$$
this shows that up to normalization,

\[ K_0(x, y) = \left( \frac{g(y)}{g(x)} \right)^{1/4}. \]

Proceeding inductively, we then get the conditions

\[ 4(j - m)(r K'_j + \{ j + \frac{1}{4} r G' \} K_j) = (\Box + H) K_{j-1}, \quad 0 < j < m. \]

When we reach the \(\sigma^{-1}\) coefficient in \((\Box + H)(\sigma^{-m} U + (\log \sigma)V)\), we get no condition on \(K_m\), but rather

\[ 2(2r L'_0 + \{ n - 2 + \frac{1}{4} r G' \} L_0) = (\Box + H) K_{m-1}. \]

Treating the coefficients in the log \(\sigma\) series inductively, we get

\[ 2k(2r L'_k + \{ n + 2k - 2 + \frac{1}{4} r G' \} L_k) = (\Box + H) L_{k-1}, \quad k \geq 1. \]

Treating the \(L_k\) as known arbitrary functions, \(K_m\) arbitrarily, we then “clean up” by solving the equations

\[ 4k(r K'_{k+m} + \{ k + m + \frac{1}{4} r G' \} K_{k+m}) = (\Box + H) K_{k+m-1} - 2(2r L'_k + \{ n + 4k - 2 + \frac{1}{4} r G' \} L_k). \]

Through this, the ambiguity in \(K_m\) propagates to the expansion of an arbitrary harmonic summand; this is of course the expected non-uniqueness.

Several ordinary differential equations of the form \(ru' + bu = f\), \(b(r) = b_0 + O(r^2)\), appear in the above discussion. If \(b_0 > 0\), solutions of the homogeneous equation \(ru' + bu = 0\) are singular at \(r = 0\), so a nonsingular solution of \(ru' + bu = f\), if any, will be unique. Setting \(u = r^{-b_0} y\), we get

\[ y' + \beta y = r^{b_0-1} f, \quad \beta := r^{-1}(b - b_0) = O(r). \]

To avoid a singularity in \(u\) at \(r = 0\), we must take \(y(0) = 0\), so that

\[ y(r) = \exp \left( - \int_0^r \beta(s) ds \right) \int_0^r \exp \left( \int_0^s \beta(z) dz \right) s^{b_0-1} f(s) ds. \]

If \(f\) is nonsingular at \(r = 0\) and \(b_0\) is a positive integer, then \(y(r) = O(r^{b_0})\). Thus \(u\) is nonsingular at \(r = 0\). Since \(n\) is even, inspection of the process which produces the functions \(K_j\) an \(L_k\) shows that they are uniquely determined and nonsingular at \(r = 0\).

When one actually tries to calculate the local invariants in the Ha \(\mathfrak{h}\) \(\mathfrak{h}\) expansion, attention quickly turns to the Taylor expansions of the Ha \(\mathfrak{h}\) \(\mathfrak{h}\) coefficients. The \(K_j\) an \(L_k\) do not have trace free Taylor coefficients; thus these coefficients do not just come from the list of \(U_{j,\ell}\) an \(V_{k,\ell}\) by our power series construction. However, the Taylor coefficients \(L_{k,p}\) of the \(L_k\) can be computed from the list \(V_{k,\ell}\) an \(vice versa\); similarly for the lists \(K_{j,p}\) an \(U_{j,\ell}\). The computation of any given entry from one list involves only finitely many entries from the counterpart list. The separation of \(V_k\) into the various \(V_{k,\ell}\) can be carried out effectively using \((8,9)\); that is, by taking the eigenresolution of \(\sigma D\) on the space of \(k\) homogeneous polynomials. The ambiguity in the definition of the \(U_{j,m}\) (resp. \(K_{m,p}\)) affects only the \(U_{j,q}\) (resp. the \(K_{q,p}\)) for \(q \geq m\), an \(\mathfrak{h}\) has no effect on the logarithmic terms.
REFERENCES


TB: Department of Mathematics, The University of Iowa, Iowa City IA 52242 USA
E-mail address: branson@math.uiowa.edu

GÓ: Department of Mathematics, Louisiana State University, Baton Rouge LA 70803 USA
E-mail address: olafsson@marais.math.lsu.edu