Spherical Representations and Mixed Symmetric Spaces

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Introduction

Let \((\mathfrak{g}, \tau)\) be a real symmetric Lie algebra and \(\mathfrak{g} = \mathfrak{h} + \mathfrak{q}\) the corresponding eigenspace decomposition for \(\tau\). We call an element \(X \in \mathfrak{q}\) hyperbolic if the operator \(\text{ad} \, X\) is diagonalizable over \(\mathbb{R}\). The existence of “enough” hyperbolic elements in \(\mathfrak{q}\) is important in many contexts. For Cartan decompositions it is crucial for the restricted root decomposition of semisimple real Lie algebras, and hence for the whole structure theory of these algebras. If \((\mathfrak{g}, \tau)\) is a non-compactly causal symmetric (NCC) Lie algebra in the sense of [HÖ96], then \(\mathfrak{q}\) contains open convex cones which are invariant under the group \(\text{Im}_\tau(\mathfrak{h})\) of inner automorphisms of \(\mathfrak{g}\) generated by \(e^{it \mathfrak{h}}\) and which consist entirely of hyperbolic elements. In the last years this class of reductive symmetric Lie algebras and the associated symmetric spaces have become a topic of very active research spreading in more and more areas. For a survey of the state of the art we refer to [HÖ96] and the literature cited there.

On the other hand there have been attempts to push this theory further to symmetric Lie algebras which are not necessarily semisimple or reductive. The simplest type (called the complex type) is where \(\mathfrak{g} = \mathfrak{h}_\mathbb{C}\) is a complexification and \(\tau\) is complex conjugation. Among these symmetric Lie algebras those for which \(\mathfrak{h}\) contains an open invariant convex cone \(W\) consisting of elliptic elements play a crucial role (cf. [Ne94a], [Ne96a], [Ne96b]). Then \(iW \subseteq \mathfrak{q} = i\mathfrak{h}\) is an open cone consisting of hyperbolic elements so that, in the special case of reductive Lie algebras, we obtain on the one hand the non-compactly causal spaces of complex type and, if we allow \(W = \mathfrak{h}\), also the Riemannian symmetric spaces coming from Cartan involutions of complex semisimple Lie algebras. For the associated symmetric spaces of complex type and the reductive spaces mentioned above one nowadays has a well developed picture of the harmonic analysis (holomorphic representations: [Ne94b], [Ne95]; spherical functions [FHO94], [HrNe96]; Hardy spaces [HÖ91], [Kr97]) and the invariant complex analysis (invariant Stein domains and plurisubharmonic functions [Ne96b]).

The first step in this program, i.e., the description of an appropriate class of not necessarily reductive symmetric Lie algebras which is general enough to incorporate all the cases mentioned above such as the mixed complex type case, the non-compactly causal spaces, and also the Riemannian symmetric spaces has been carried out in [KN96], which we will use as a reference for the basic structure theory and convex geometry of mixed, i.e., non-reductive, symmetric Lie algebras.

The next step that we carry out in this paper is the description of the structure of the associated global objects such as complex domains which are curved analogs of tube domains over convex cones. Furthermore we investigate the general representation theory and explain how certain representations can be realized in spaces of holomorphic functions on the aforementioned domains.

In Section I we collect the notation and the facts from [KN96] we shall need throughout this paper. In Section II we then turn to product decompositions of the corresponding groups.
These decompositions have various applications such as integration formulas and trivializations of certain holomorphic vector bundles. In this section many proofs consist in reducing everything to the case of reductive or simple Lie algebras which is completely discussed in [HO96]. As a first application of the decomposition theorems we explain how one can construct on certain domains acted on by $H$ holomorphic vector valued functions which are $H$-eigenfunctions for a prescribed character. These functions play a key role in the realization theory of spherical representations.

Section III contains the description of various semigroups related to symmetric spaces. Basically these semigroups are of the type $\Gamma_H(C) = H \exp(C)$, where $G/H$ is a symmetric space, and $C \subseteq \mathfrak{g}$ is a weakly hyperbolic $H$-invariant cone. Such semigroups arise naturally if $G/H$ carries an invariant non-compactly causal structure ([HO96]). On the other hand the polar decomposition of $\Gamma_H(C)$ is quite similar to the Cartan decomposition of a real semisimple Lie group. We also describe a complex version of these semigroups and how they fit into the product decompositions discussed in Section II.

To each semigroup $\Gamma_H(C)$ we can associate the domain $\text{Exp}(C) \subseteq G/H$ which should be thought of as the future of the base point. In Section IV we construct a complex domain $\Xi(C)$ which is the curved analog of a tube domain over the cone $C$. It contains the dual symmetric space $G^c/H$ in its boundary (it is a sort of Shilov boundary for $\Xi(C)$), and on the other hand it has the structure of an associated bundle of the type $G^c \times_H C$. The key point in Section IV is to clarify the interplay between the complex analysis of $\Xi(C)$ and its bundle structure which is not in any obvious way related to its complex geometry.

In Section V we then explain which Hilbert spaces carrying representation of $\Gamma_H(C)$ can be realized in an equivariant way as spaces of holomorphic functions on $\Xi(C)$ and that the corresponding representations can be characterized as spherical representations of some sort. An important tool for these realizations is a result describing holomorphic functions on $\Xi(C)$ as the set of all holomorphic functions on a complex semigroup which are invariant under the group $H$.

In the last section we then turn to irreducible spherical representations. We explain in which sense they are highest weight representations of the dual Lie algebra $\mathfrak{g}^c = \mathfrak{h} + i \mathfrak{q}$ and give a necessary condition for a unitary highest weight representation to correspond to a spherical representation.

One of the next steps in the investigation of spherical representations of mixed symmetric spaces is to classify all irreducible spherical unitary highest weight representations. Since this involves a quite detailed analysis of certain singular highest weight representations and it is not even carried out for irreducible spaces, the solution of this problem seems to be quite complicated.

Another project building on this paper will be the construction of Hardy spaces on the domains $\Xi(C)$ for general groups. For non-reductive groups this construction is more involved because the symmetric space $G^c/H$ does not always possess a $G^c$-invariant measure. Nevertheless it seems to be possible to construct analogs of Hardy spaces on $\Xi(C)$.

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